MODEL ANSWERS TO THE FIRST HOMEWORK

1. Applying Example 2.2 with $a = 1$, we get

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2+1)^2} = \pi.$$

As the integrand is even this means

$$\int_{0}^{\infty} \frac{\mathrm{d}x}{(x^2+1)^2} = \frac{\pi}{2}.$$

2. We calculate

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^4+1}$$

and divide by 2. We integrate over the standard contour γ_1 from $-R$ to R and the semicircle of radius R centred at 0 in the upper half plane, γ_2. The integrand is

$$f(z) = \frac{1}{z^4+1}.$$

This has isolated singularities at the zeroes of $z^4 + 1$. This is the same as the fourth roots of -1, which is the same as the eighth roots of unity, which are not fourth roots of unity. So the singularities of $f(z)$ are at

$$e^{i\pi/4} \qquad e^{3i\pi/4} \qquad e^{5i\pi/4} \qquad \text{and} \qquad e^{7i\pi/4}.$$

Of these, only the first two are in the upper half plane (in fact, since these are roots of a real polynomial the roots come in complex conjugate pairs; two roots are above the real axis and two roots are below).
We compute the residues at these two points. Both of them are simple poles. We have

$$\begin{aligned}
\operatorname{Res}_{e^{i\pi/4}} f(z) &= \lim_{z \to e^{i\pi/4}} \frac{(z - e^{i\pi/4})}{z^4+1} \\
&= \lim_{z \to e^{i\pi/4}} \frac{1}{4z^3} \\
&= \frac{1}{4e^{3i\pi/4}} \\
&= \frac{1}{4} e^{-3i\pi/4}.
\end{aligned}$$

By contrast

$$\begin{aligned}\mathrm{Res}_{e^{3i\pi/4}} f(z) &= \lim_{z\to e^{3i\pi/4}} \frac{(z-e^{3i\pi/4})}{z^4+1}\\ &= \lim_{z\to e^{3i\pi/4}} \frac{1}{4z^3}\\ &= \frac{1}{4e^{9i\pi/4}}\\ &= \frac{1}{4e^{i\pi/4}}\\ &= \frac{1}{4}e^{-i\pi/4}.\end{aligned}$$

The residue theorem implies that

$$\begin{aligned}\int_\gamma \frac{\mathrm{d}z}{z^4+1} &= 2\pi i(\mathrm{Res}_{e^{i\pi/4}} f(z) + \mathrm{Res}_{e^{3i\pi/4}} f(z))\\ &= 2\pi i\left(\frac{1}{4}e^{-3i\pi/4} + \frac{1}{4}e^{-i\pi/4}\right)\\ &= \frac{\pi i}{2}\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\\ &= \frac{\pi}{\sqrt{2}}.\end{aligned}$$

As the improper integral

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^4+1}$$

converges, the integral over the interval $[-R, R]$ converges to the integral we want to compute, in the usual way.
The length L of the semicircle is πR and we have to estimate the maximum value M of $|f(z)|$ over the semicircle

$$\begin{aligned}|f(z)| &= \left|\frac{1}{z^4+1}\right|\\ &= \frac{1}{|z^4+1|}\\ &\leq \frac{1}{R^4-1}.\end{aligned}$$

It follows that

$$\begin{aligned}\left|\int_{\gamma_2} \frac{\mathrm{d}z}{z^4-1}\right| &\leq LM \\ &\leq \frac{\pi R}{R^4-1}.\end{aligned}$$

The key point is that this rational fraction is bottom heavy, so that as R goes to infinity the rational function goes to zero. It follows that

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^4+1} = \frac{\pi}{\sqrt{2}}.$$

Therefore

$$\int_{0}^{\infty} \frac{\mathrm{d}x}{x^4+1} = \frac{\pi}{2\sqrt{2}}.$$

3. We calculate

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^6+1}$$

and divide by 2. We integrate over the standard contour γ_1 from $-R$ to R and the semicircle of radius R centred at 0 in the upper half plane, γ_2. The integrand is

$$f(z) = \frac{1}{z^6+1}.$$

This has isolated singularities at the zeroes of z^6+1. This is the same as the sixth roots of -1, which is the same as the twelfth roots of unity, which are not sixth roots of unity. So the singularities of $f(z)$ in the upper half plane are located at

$$e^{i\pi/6} \qquad e^{3i\pi/6} = e^{i\pi/2} \qquad \text{and} \qquad e^{5i\pi/6}.$$

The other three isolated singularities are in the lower half plane.
We compute the residues at these three points. All three of them are simple poles. We have

$$\begin{aligned}\operatorname{Res}_{e^{i\pi/6}} f(z) &= \lim_{z\to e^{i\pi/6}} \frac{(z-e^{i\pi/6})}{z^6+1} \\ &= \lim_{z\to e^{i\pi/4}} \frac{1}{6z^5} \\ &= \frac{1}{6e^{5i\pi/6}} \\ &= \frac{1}{6}e^{-5i\pi/6}.\end{aligned}$$

We also have

$$\begin{aligned}\operatorname{Res}_{e^{i\pi/2}} f(z) &= \frac{1}{6e^{5i\pi/2}}\\ &= \frac{1}{6}e^{-5i\pi/2}\\ &= \frac{1}{6}e^{-i\pi/2},\end{aligned}$$

and

$$\begin{aligned}\operatorname{Res}_{e^{5i\pi/6}} f(z) &= \frac{1}{6e^{25i\pi/6}}\\ &= \frac{1}{6}e^{-25i\pi/6}\\ &= \frac{1}{6}e^{-i\pi/6}.\end{aligned}$$

The residue theorem implies that

$$\begin{aligned}\int_\gamma \frac{\mathrm{d}z}{z^6+1} &= 2\pi i(\operatorname{Res}_{e^{i\pi/6}} f(z) + \operatorname{Res}_{e^{i\pi/2}} f(z) + \operatorname{Res}_{e^{5i\pi/6}} f(z))\\ &= 2\pi i\left(\frac{1}{6}e^{-5i\pi/6} + \frac{1}{6}e^{-i\pi/2} + \frac{1}{6}e^{-i\pi/6}\right)\\ &= \frac{\pi i}{3}\left(-\frac{\sqrt{3}}{2} - \frac{i}{2} - i + \frac{\sqrt{3}}{2} - \frac{i}{2}\right)\\ &= \frac{2\pi}{3}.\end{aligned}$$

As the improper integral

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^6+1}$$

converges, the integral over the interval $[-R, R]$ converges to the integral we want to compute, in the usual way.

The length L of the semicircle is πR and we have to estimate the maximum value M of $|f(z)|$ over the semicircle

$$\begin{aligned}|f(z)| &= \left|\frac{1}{z^6+1}\right|\\ &= \frac{1}{|z^6+1|}\\ &\leq \frac{1}{R^6-1}.\end{aligned}$$

It follows that

$$\begin{aligned}\left|\int_{\gamma_2} \frac{\mathrm{d}z}{z^6+1}\right| &\leq LM \\ &\leq \frac{\pi R}{R^6-1}.\end{aligned}$$

As this rational fraction is bottom heavy, the rational function goes to zero, as R goes to infinity. It follows that

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^6+1} = \frac{2\pi}{3}.$$

Hence

$$\int_{0}^{\infty} \frac{\mathrm{d}x}{x^6+1} = \frac{\pi}{3}.$$

4. We integrate over the standard contour γ_1 from $-R$ to R and the semicircle of radius R centred at 0 in the upper half plane, γ_2. The integrand is

$$f(z) = \frac{1}{z^2+2z+2}.$$

This has isolated singularities at the zeroes of z^2+2z+2. As this is a real polynomial, one zero is in the upper half plane, one in the lower. If we complete the square we want to solve

$$(z+1)^2+1=0 \qquad \text{so that} \qquad z=-1\pm i.$$

The singularity in the upper half plane is $-1+i$.
As this is a simple zero we have

$$\begin{aligned}\operatorname{Res}_{-1+i} f(z) &= \lim_{z\to -1+i} \frac{(z+1-i)}{z^2+2z+1} \\ &= \lim_{z\to -1+i} \frac{1}{2(z+1)} \\ &= \frac{1}{2i} \\ &= -\frac{i}{2}.\end{aligned}$$

The residue theorem implies that

$$\begin{aligned}\int_{\gamma} \frac{\mathrm{d}z}{z^2+2z+1} &= 2\pi i \operatorname{Res}_{-1+i} f(z) \\ &= 2\pi i \cdot -\frac{i}{2} \\ &= \pi.\end{aligned}$$

As the improper integral

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2 + 2x + 2}$$

converges, the integral over the interval $[-R, R]$ converges to the integral we want to compute, in the usual way.
The length L of the semicircle is πR and we have to estimate the maximum value M of $|f(z)|$ over the semicircle

$$\begin{aligned} |f(z)| &= \left| \frac{1}{z^2 + 2z + 1} \right| \\ &= \frac{1}{|z^2 + 2z + 1|} \\ &\leq \frac{1}{R^2 - 2R - 1}. \end{aligned}$$

It follows that

$$\begin{aligned} \left| \int_{\gamma_2} \frac{\mathrm{d}z}{z^6 + 1} \right| &\leq LM \\ &\leq \frac{\pi R}{R^2 - 2R - 1}. \end{aligned}$$

As this rational fraction is bottom heavy, the rational function goes to zero, as R goes to infinity. It follows that

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2 + 2x + 2} = \pi.$$

5. Let $\gamma = \gamma_1 + \gamma_2 + \gamma_3$ be the contour consisting of γ_1 the line segment from 0 to R, γ_2 the arc of the circle from R to $Re^{2\pi i/3}$ and γ_3 the line segment from $Re^{2\pi i/3}$ to 0.
If U is the intersection of the angular sector between 0 and $2\pi/3$ and the circle of radius R centred at the origin

$$U = \{\, z \in \mathbb{C} \,|\, 0 < \operatorname{Arg}(z) < 2\pi/3, 0 < |z| < R \,\}$$

then the boundary of U is γ.
Let

$$f(z) = \frac{1}{z^3 + 1}.$$

The poles of $f(z)$ are located at the cube roots of -1. These are the sixth roots of unity which are not cube roots of unity:

$$e^{\pi i/3}; \qquad e^{3\pi i/3} = e^{\pi i} \qquad \text{and} \qquad e^{5\pi i/3}.$$

Of these only the first belongs to U (assuming $R > 1$).

All of these are simple singularities. We have

$$\begin{aligned}
\operatorname{Res}_{e^{\pi i/3}} f(z) &= \lim_{z \to e^{i\pi/3}} \frac{z - e^{i\pi/3}}{z^3 + 1} \\
&= \lim_{z \to e^{i\pi/3}} \frac{1}{3z^2} \\
&= \frac{1}{3e^{2i\pi/3}} \\
&= \frac{e^{-2i\pi/3}}{3}.
\end{aligned}$$

The residue theorem implies that

$$\begin{aligned}
\int_\gamma \frac{\mathrm{d}z}{z^3 + 1} &= 2\pi i \operatorname{Res}_{e^{\pi i/3}} f(z) \\
&= \frac{2\pi i}{3} e^{-2i\pi/3}.
\end{aligned}$$

As

$$I = \int_0^\infty \frac{\mathrm{d}x}{x^3 + 1},$$

is an improper integral which converges, the integral over γ_1 approaches this integral as R goes to infinity:

$$\begin{aligned}
\lim_{R \to \infty} \int_{\gamma_1} \frac{\mathrm{d}z}{z^3 + 1} &= \lim_{R \to \infty} \int_0^R \frac{\mathrm{d}x}{x^3 + 1} \\
&= \int_0^\infty \frac{\mathrm{d}x}{x^3 + 1} \\
&= I.
\end{aligned}$$

We estimate the integral over γ_2. The length L of γ_2 is $2\pi R/3$. The maximum value M of $|f(z)|$ is

$$\begin{aligned}
|f(z)| &= \left| \frac{1}{z^3 + 1} \right| \\
&= \frac{1}{|z^3 + 1|} \\
&\leq \frac{1}{R^3 - 1}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\left| \int_{\gamma_2} \frac{\mathrm{d}z}{z^3 + 1} \right| &\leq LM \\
& \frac{2\pi R}{3(R^2 - 1)}.
\end{aligned}$$

As this goes to zero as R goes to infinity, it follows that the integral over γ_2 goes to zero.
For the integral over γ_3 we use the parameterisation

$$\gamma_3(t) = (R-t)e^{2\pi i/3} \qquad \text{where} \qquad t \in [0, R].$$

We have

$$\begin{aligned}\lim_{R\to\infty} \int_{\gamma_3} \frac{\mathrm{d}z}{z^3+1} &= -e^{2\pi i/3} \lim_{R\to\infty} \int_0^R \frac{\mathrm{d}t}{t^3+1} \\ &= -e^{2\pi i/3} \int_0^\infty \frac{\mathrm{d}x}{x^3+1} \\ &= -e^{2\pi i/3} I.\end{aligned}$$

It follows that

$$\begin{aligned}\frac{2\pi i}{3} e^{-2i\pi/3} &= I - e^{2\pi i/3} I \\ &= (1 - e^{2\pi i/3}) I.\end{aligned}$$

Let

$$\omega = e^{2\pi i/3} \qquad \text{so that} \qquad \omega^3 = 1.$$

We have

$$\begin{aligned}\int_0^\infty \frac{\mathrm{d}x}{x^3+1} &= I \\ &= \frac{2\pi i}{3} \frac{e^{-2i\pi/3}}{1 - e^{2\pi i/3}} \\ &= \frac{2\pi i}{3} \frac{\omega^2}{1-\omega} \\ &= \frac{2\pi i}{3} \frac{\omega^3}{\omega - \omega^2} \\ &= \frac{2\pi i}{3} \frac{1}{i\sqrt{3}} \\ &= \frac{2\pi}{3\sqrt{3}}.\end{aligned}$$

6. We use the standard contour and we integrate

$$f(z) = \frac{z e^{iz}}{(z^2+1)(z^2+4)}.$$

This has poles at $\pm i$ and $\pm 2i$. All of these are simple poles and i and $2i$ are the only singularities in the upper half plane.

We have

$$\begin{aligned}
\operatorname{Res}_i f(z) &= \lim_{z\to i} \frac{(z-i)ze^{iz}}{(z^2+1)(z^2+4)} \\
&= \lim_{z\to i} \frac{ze^{iz}}{(z+i)(z^2+4)} \\
&= \frac{ie^{-1}}{(i+i)(-1+4)} \\
&= \frac{1}{6e}.
\end{aligned}$$

We also have

$$\begin{aligned}
\operatorname{Res}_{2i} f(z) &= \lim_{z\to 2i} \frac{(z-2i)ze^{iz}}{(z^2+1)(z^2+4)} \\
&= \lim_{z\to 2i} \frac{ze^{iz}}{(z^2+1)(z+2i)} \\
&= \frac{2ie^{-2}}{(-3)(4i)} \\
&= -\frac{1}{6e^2}.
\end{aligned}$$

The residue theorem implies

$$\begin{aligned}
\int_\gamma \frac{z^{iz}\,\mathrm{d}z}{(z^2+1)(z^2+4)} &= 2\pi i\left(\operatorname{Res}_i f(z) + \operatorname{Res}_{2i} f(z)\right) \\
&= \frac{\pi i}{3}\frac{e-1}{e^2}.
\end{aligned}$$

The improper integral

$$\int_{-\infty}^{\infty} \frac{xe^{ix}\,\mathrm{d}x}{(x^2+1)(x^2+4)}$$

converges, since the absolute value of the integrand

$$\frac{x^{ix}}{(x^2+1)(x^2+4)}$$

looks like $1/x^3$ when x is large and the integral of $1/x^3$ converges. Taking the limit as R goes to ∞ it follows that the integral over γ_1 of $f(z)$ converges to the integral we are after.

For the integral over γ_2 we have to estimate M:

$$\begin{aligned}|f(z)| &= \left|\frac{ze^{iz}}{(z^2+1)(z^2+4)}\right| \\ &= \frac{|ze^{iz}|}{|(z^2+1)(z^2+4)|} \\ &\leq \frac{R}{(R^2-1)(R^2-4)}.\end{aligned}$$

It follows that

$$\begin{aligned}\left|\int_\gamma \frac{ze^{iz}\,\mathrm{d}z}{(z^2+1)(z^2+4)}\right| &\leq LM \\ &\leq \frac{\pi R^2}{(R^2-1)(R^2-4)},\end{aligned}$$

which goes to zero as R goes to infinity.
It follows that

$$\int_{-\infty}^{\infty} \frac{xe^{ix}\,\mathrm{d}x}{(x^2+1)(x^2+4)} = \frac{\pi i}{3}\frac{e-1}{e^2}.$$

Taking the imaginary parts gives

$$\int_{-\infty}^{\infty} \frac{x\sin x\,\mathrm{d}x}{(x^2+1)(x^2+4)} = \frac{\pi}{3}\frac{e-1}{e^2}.$$

7. We integrate

$$f(z) = \frac{e^{iz}}{z^2+4z+5}$$

over the standard contour. The singularities of $f(z)$ are at the zeroes of z^2+4z+5. We have

$$z^2+4z+5 = (z+2)^2+1.$$

Hence the roots of z^2+4z+5 are at

$$-2 \pm i.$$

Both are simple poles and of these $-2+i$ is in the upper half plane. The residue at $-2+i$ is

$$\begin{aligned}\operatorname{Res}_{-2+i} f(z) &= \lim_{z\to -2+i}(z+2-i)\frac{e^{iz}}{z^2+4z+5}\\ &= \lim_{z\to -2+i}\frac{e^{iz}}{(z+2+i)}\\ &= \frac{e^{-1-2i}}{2i}\end{aligned}$$

The residue theorem implies that

$$\begin{aligned}\int_\gamma \frac{e^{iz}\,\mathrm{d}z}{z^2+4z+5} &= 2\pi i\frac{e^{-1-2i}}{2i}\\ &= \pi e^{-1-2i}.\end{aligned}$$

As the improper integral

$$\int_{-\infty}^{\infty}\frac{e^{ix}\,\mathrm{d}x}{x^2+4x+5}$$

converges the integral over γ_1 tends to this integral as we let R go infinity.
For the integral over γ_2, we have

$$\begin{aligned}|f(z)| &= \left|\frac{e^{iz}}{z^2+4z+5}\right|\\ &= \frac{|e^{iz}|}{|z^2+4z+5|}\\ &\leq \frac{1}{R^2-4R-5}.\end{aligned}$$

If follows that

$$\begin{aligned}\left|\int_{-\infty}^{\infty}\frac{e^{iz}\,\mathrm{d}z}{z^2+4z+5}\right| &\leq LM\\ &\leq \frac{\pi R}{R^2-4R-5},\end{aligned}$$

which goes to zero as R goes to infinity.
Hence

$$\int_{-\infty}^{\infty}\frac{e^{ix}\,\mathrm{d}x}{x^2+4x+5} = \pi e^{-1-2i}.$$

To finish we just need to take the imaginary part of both sides. We have

$$\begin{aligned}\pi e^{-1-2i} &= \frac{\pi}{e} e^{-2i} \\ &= \frac{\pi}{e}(\cos 2 - i \sin 2).\end{aligned}$$

Therefore

$$\int_{-\infty}^{\infty} \frac{\sin x \, \mathrm{d}x}{x^2 + 4x + 5} = -\frac{\pi}{e} \sin 2.$$

8. We integrate

$$f(z) = \frac{e^{iaz}}{(z^2 + b^2)^2}$$

over the standard contour. As the improper integral

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{(x^2 + b^2)^2} \, \mathrm{d}x$$

converges, we have

$$\begin{aligned}\lim_{R \to \infty} \int_{\gamma_1} \frac{e^{iaz}}{(z^2 + b^2)^2} \, \mathrm{d}z &= \lim_{R \to \infty} \int_{-R}^{R} \frac{e^{iax}}{(x^2 + b^2)^2} \, \mathrm{d}x \\ &= \int_{-\infty}^{\infty} \frac{e^{iax}}{(x^2 + b^2)^2} \, \mathrm{d}x.\end{aligned}$$

For the integral over γ_2 we have

$$\begin{aligned}|f(z)| &= \frac{|e^{iaz}|}{|z^2 + b^2|^2} \\ &\leq \frac{1}{(R^2 - b^2)^2}.\end{aligned}$$

Hence

$$\begin{aligned}\left| \int_{\gamma_2} \frac{e^{iaz}}{(z^2 + b^2)^2} \, \mathrm{d}z \right| &\leq LM \\ &\leq \frac{\pi R}{(R^2 - b^2)^2},\end{aligned}$$

which goes to zero, as R goes to infinity.

$f(z)$ has isolated singularities at $\pm ib$ and only ib belongs to the upper half plane. As this is a double pole we have

$$\begin{aligned}
\operatorname{Res}_{ib} f(z) &= \lim_{z \to ib} \frac{\mathrm{d}}{\mathrm{d}z} (z - ib)^2 \frac{e^{iaz}}{(z^2 + b^2)^2} \\
&= \lim_{z \to ib} \frac{\mathrm{d}}{\mathrm{d}z} \frac{e^{iaz}}{(z + ib)^2} \\
&= \lim_{z \to ib} \frac{iae^{iaz}(z + ib)^2 - 2e^{iaz}(z + ib)}{(z + ib)^4} \\
&= \lim_{z \to ib} \frac{ia(z + ib) - 2}{(z + ib)^3} e^{iaz} \\
&= \frac{ia(2ib) - 2}{(2ib)^3} e^{-ab} \\
&= -i \frac{ab + 1}{4b^3} e^{-ab}.
\end{aligned}$$

The residue theorem implies that

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^{iax}}{(x^2 + b)^2} \, \mathrm{d}x &= 2\pi i \operatorname{Res}_{ib} f(z) \\
&= 2\pi i \cdot -i \frac{ab + 1}{4b^3} e^{-ab} \\
&= \frac{2\pi}{4b^3} (1 + ab) e^{-ab}.
\end{aligned}$$

Taking the real part gives

$$\int_{0}^{\infty} \frac{\cos ax}{(x^2 + b)^2} \, \mathrm{d}x = \frac{2\pi}{4b^3} (1 + ab) e^{-ab}.$$

Finally, using the fact $\cos ax$ is even we get

$$\int_{0}^{\infty} \frac{\cos ax}{(x^2 + b)^2} \, \mathrm{d}x = \frac{\pi}{4b^3} (1 + ab) e^{-ab}.$$

9. We integrate

$$f(z) = \frac{e^{iaz}}{(z^2 + b^2)^2}$$

over the standard contour. As the improper integral

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{(x^2 + b^2)^2} \, \mathrm{d}x$$

converges, we have

$$\lim_{R\to\infty}\int_{\gamma_1}\frac{e^{iaz}}{(z^2+b^2)^2}\,\mathrm{d}z=\lim_{R\to\infty}\int_{-R}^{R}\frac{e^{iax}}{(x^2+b^2)^2}\,\mathrm{d}x$$
$$=\int_{-\infty}^{\infty}\frac{e^{iax}}{(x^2+b^2)^2}\,\mathrm{d}x.$$

For the integral over γ_2 we have

$$|f(z)|=\frac{|e^{iaz}|}{|z^2+b^2|^2}$$
$$\le\frac{1}{(R^2-b^2)^2}.$$

Hence

$$\left|\int_{\gamma_2}\frac{e^{iaz}}{(z^2+b^2)^2}\,\mathrm{d}z\right|\le LM$$
$$\le\frac{\pi R}{(R^2-b^2)^2},$$

which goes to zero, as R goes to infinity.
$f(z)$ has isolated singularities at $\pm ib$ and only ib belongs to the upper half plane. As this is a double pole we have

$$\mathrm{Res}_{ib}\,f(z)=\lim_{z\to ib}\frac{\mathrm{d}}{\mathrm{d}z}(z-ib)^2\frac{e^{iaz}}{(z^2+b^2)^2}$$
$$=\lim_{z\to ib}\frac{\mathrm{d}}{\mathrm{d}z}\frac{e^{iaz}}{(z+ib)^2}$$
$$=\lim_{z\to ib}\frac{iae^{iaz}(z+ib)^2-2e^{iaz}(z+ib)}{(z+ib)^4}$$
$$=\lim_{z\to ib}\frac{ia(z+ib)-2}{(z+ib)^3}e^{iaz}$$
$$=\frac{ia(2ib)-2}{(2ib)^3}e^{-ab}$$
$$=-i\frac{ab+1}{4b^3}e^{-ab}.$$

The residue theorem implies that

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{(x^2+b)^2}\,\mathrm{d}x = 2\pi i\,\mathrm{Res}_{ib} f(z)$$
$$= 2\pi i \cdot -i\frac{ab+1}{4b^3}e^{-ab}$$
$$= \frac{2\pi}{4b^3}(1+ab)e^{-ab}.$$

Taking the real part gives

$$\int_{0}^{\infty} \frac{\cos ax}{(x^2+b)^2}\,\mathrm{d}x = \frac{2\pi}{4b^3}(1+ab)e^{-ab}.$$

Finally, using the fact $\cos ax$ is even we get

$$\int_{0}^{\infty} \frac{\cos ax}{(x^2+b)^2}\,\mathrm{d}x = \frac{\pi}{4b^3}(1+ab)e^{-ab}.$$

Challenge Problems: (Just for fun)

10. We are going to use the answers to 2 and 3 as a guide to how to solve this problem. We do the usual things; integrate over the usual contour, argue that as R goes to infinity the integral over γ_1 goes to twice the integral we want and the integral over the semircircle γ_2 goes to zero, as the absolute value of the integral is bounded above by

$$\frac{\pi R^{2m+1}}{R^{2n}-1}.$$

Putting all of this together we get

$$\int_{0}^{\infty} \frac{x^{2m}}{x^{2n}+1}\,\mathrm{d}x = \frac{1}{2}2\pi i\sum_{i=1}^{n}\mathrm{Res}_{a_i} f(z) \qquad \text{where} \qquad f(z) = \frac{z^{2m}}{z^{2n}-1}.$$

The hard part is to compute the sum of the residues. The singularities of $f(z)$ are located at the roots of $z^{2n}+1$ in the upper half plane. The roots of $z^{2n}+1$ are $2n$th roots of -1. These are the $4n$th roots of unity which are not $2n$th roots of unity. A $4n$th root of unity is of the form

$$e^{2\pi ik/4n} = e^{\pi ik/2n}$$

where k is an integer. If this is not a $2n$th root of unity we should take k odd. To achieve this, simply replace k by $2k-1$. The singularities in the upper half plane have argument between 0 and π. This means

$$1 \leq 2k-1 \leq 2n.$$

It follows that the isolated singularities of $f(z)$ in the upper half plane are

$$a_k = e^{\pi i(2k-1)/2n} \qquad \text{where} \qquad 1 \leq k \leq n.$$

All of these singularities are simple. Let us compute the residue of $f(z)$ at a_k:

$$\begin{aligned}
\text{Res}_{a_k} f(z) &= \lim_{z \to a_k} (z - a_k) f(z) \\
&= \lim_{z \to a_k} \frac{(z - a_k) z^{2m}}{z^{2n} + 1} \\
&= \lim_{z \to a_k} \frac{z^{2m} + 2m(z - a_k) z^{2m-1}}{2n z^{2n-1}} \\
&= \frac{a_k^{2m}}{2n a_k^{2n-1}} \\
&= \frac{a_k^{2m+1-2n}}{2n} \\
&= -\frac{a_k^{2m+1}}{2n} \\
&= -\frac{e^{\pi i(2k-1)(2m+1)/2n}}{2n}.
\end{aligned}$$

The sum of the residues is then the sum of a geometric series

$$-\frac{e^{\pi i(2m+1)/2n} - e^{\pi i(2n+1)(2m+1)/2n}}{2n(1 - e^{\pi i(2m+1)/n})}.$$

We have

$$\begin{aligned}
-\frac{e^{\pi i(2m+1)/2n} - e^{\pi i(2n+1)(2m+1)/2n}}{2n(1 - e^{\pi i(2m+1)/n})} &= -e^{\pi i(2m+1)/2n} \frac{1 - e^{\pi i(2n(2m+1)/2n}}{2n(1 - e^{\pi i(2m+1)/n})} \\
&= -\frac{1}{n(e^{-\pi i(2m+1)/2n} - e^{\pi i(2m+1)/2n})} \\
&= \frac{1}{2n(i \sin((2m+1)/2n)} \\
&= -\frac{i}{2n} \csc((2m+1)/2n).
\end{aligned}$$