MODEL ANSWERS TO THE FOURTH HOMEWORK

1. Let

$$p(z) = z^4 + 2z^2 - z + 1.$$

Consider

$$p(x) = x^4 + 2x^2 - x + 1.$$

If $x \leq 0$ then $p(x) > 0$, since every term is non-negative and the constant term is positive. If $x \in [0, 1)$ then

$$1 - x > 0 \qquad \text{so that} \qquad p(x) > 0.$$

If $x \geq 1$ then

$$2x^2 - x > 0 \qquad \text{so that} \qquad p(x) > 0.$$

It follows that $p(z)$ has no real roots. Since its coefficients are real, it roots come in complex conjugate pairs and so two roots are in the upper half plane and two roots are in the lower half plane. Further it suffices to prove that one quadrant has one root.
Consider what happens in the first quadrant. We go along the boundary of the quarter circle in the first quadrant of radius R centred at the origin,

$$\gamma = \gamma_1 + \gamma_2 + \gamma_3,$$

and consider the change in the argument, when R is large.
Over γ_1 we go from 0 to R so that $p(z)$ is always real and the argument is constant. Over γ_2 the dominant term is z^4 and so the change in the argument is roughly

$$4\frac{\pi}{2} = 2\pi.$$

On γ_3 we have

$$z = iy \qquad \text{where} \qquad y \in [0, R].$$

In this case

$$p(iy) = y^4 - 2y^2 + 1 - iy$$

Consider $p(iR)$. The dominant term is y^4, so we are very close to R^4. The dominant term in its imaginary part is $-y$, which is negative. So $p(iR)$ is in in the fourth quadrant and the argument is close to 2π.
As we traverse γ_3 where do we cross the real line? This is when $y = 0$. It follows that we stay in the third and fourth quadrant along γ_3. When y is close to 0 we must approach 1 from the fourth quadrant, so that the argument approaches 2π. So the change in the argument over γ_3 is close to zero.

Thus the total change in the argument is roughly 2π. Since the change in the argument is a multiple of 2π, it must be exactly 2π. But then the number of zeroes in the first quadrant is one. It follows that $z^4 + 2z^2 - z + 1$ has exactly one root in each quadrant.
2. Let
$$p(z) = z^4 + z^3 + 4z^2 + 3z + 2.$$
Consider
$$p(x) = x^4 + x^3 + 4x^2 + 3x + 2.$$
If $x \geq 0$ then all terms are positive and $p(x) > 0$. If $x \in [-1/2, 0]$ then
$$x^3 + 3x + 2 > 0 \qquad \text{and so} \qquad p(x) > 0.$$
If $x \in [-1, -1/2]$ then
$$4x^2 + 2x \geq 0 \qquad \text{and} \qquad 2 + x + x^3 \geq 0$$
and so $p(x) \geq x^4 > 0$. If $x \leq -1$ then
$$x^4 + x^3 \geq 0 \qquad \text{and} \qquad 4x^2 + 3x > 0$$
and so $p(x) > 0$.
It follows that $p(z)$ does not have any real roots. Now we determine how many roots it has in the first quadrant. We traverse the contour of Question 1 and we determine the change in the argument.
$p(z)$ is real along γ_1 and there is no change in the argument. Along γ_2 the dominant term is still z^4 and the change in the argument is approximately 2π.
Over γ_3 we have
$$p(iy) = y^4 - 4y^2 + 2 + i(-y^3 + 3y).$$
The dominant term of $p(iR)$ is R^4, so that we are close to R^4. The dominant term of the imaginary part is $-R^3$ so that $p(iR)$ belongs to the fourth quadrant.
We cross the real line when $y^3 = 3y$, so that $y = 0$, $\sqrt{3}$ and $-\sqrt{3}$. $y \geq 0$ over γ_3, so we only cross the real line once, at $\sqrt{3}$. When $y = \sqrt{3}$ the real part is
$$9 - 12 + 2 = -1.$$
So we cross the real line in the left half plane. The only possibility is that on γ_3 we go from the fourth quadrant back to the third quadrant to the second quadrant and then to the first quadrant. So the change in the argument is roughly -2π.
Therefore the change in the argument over the whole of γ is roughly zero, so that it is exactly zero, as it is a multiple of 2π. Therefore $p(z)$ has no zeroes in the first quadrant.

As $p(z)$ has real coefficients it roots come in complex conjugate pairs. As it has no roots in the first quadrant it has no roots in the fourth quadrant. As it has four roots, two roots are in the upper half plane. Therefore it has two roots in the third quadrant and two roots in the third quadrant.

3. Let

$$p(z) = z^6 + 4z^4 + z^3 + 2z^2 + z + 5.$$

Consider

$$p(x) = x^6 + 4x^4 + x^3 + 2x^2 + x + 5.$$

If $x > 0$ then all terms are non-negative and so $p(x) \geq 5 > 0$. Thus $p(z)$ has no real roots in the first quadrant.

We consider the change in the argument over the contour in Question 1. $p(z)$ is real over γ_1 and so there is no change in the argument. Over γ_2 the dominant term is z^6 and so the change in the argument is approximately

$$6\frac{\pi}{2} = 3\pi.$$

Over γ_3, $z = iy$ where $y \in [0, R]$. We have

$$p(iy) = -y^6 + 4y^4 - 2y^2 + 5 + i(-y^3 + y).$$

When $y = R$ the dominant term is $-y^6$ and so $p(iR)$ is close to $-R^6$. The dominant term of the imaginary part is $-y^3$ and so $p(iR)$ is in the third quadrant.

We cross the real axis when $y^3 = y$, so when $y = 0$, 1 and -1. So on γ_3 we cross the real axis only when $y = 1$. In this case the real part is

$$-1 + 4 - 2 + 5 > 0.$$

Over γ_3 we start in the third quadrant and we end up the the fourth quadrant up to i. The change in the argument from iR to i is roughly π. After that we return to the real axis and the change in the argument is zero.

So the total change in the argument is roughly 4π. As the change in the argument is exactly a multiple of 2π, the change in the argument is exactly 4π. But then there are two zeroes in the first quadrant.

4. Let

$$p(z) = z^4 + z^3 + 4z^2 + \alpha z + 3.$$

Let U be the intersection of the open disk of radius R with centre 0 and the left half plane. The boundary of U is

$$\gamma = \gamma_1 + \gamma_2,$$

where γ_1 is the line segment from $-iR$ to iR and γ_2 is the semicircle of radius R centred at the 0 in the left half plane.

As we go around γ_2 the dominant term is z^4, so that the change in the argument is approximately

$$4\pi.$$

Consider what happens as we go from $-iR$ to iR. If we substitute $z = iy$ then we get

$$p(iy) = y^4 - 4y^2 + 3 + i(-y^3 + \alpha y).$$

Consider $p(iR)$. The dominant term is y^4 and so we are close to R^4. The dominant imaginary term is $-y^3$ and so the imaginary part is close to $-R^3$, so that it is negative. So $p(iR)$ belongs to the fourth quadrant and the argument is close to 2π.

Now consider $p(-iR)$. The dominant term is y^4 and so we are close to R^4. The dominant imaginary term is y^3 and so the imaginary part is close to R^3, so that it is positive. So $p(-iR)$ belongs to the first quadrant and the argument is close to 0.

How many times and where do we cross the real axis? This is when the imaginary part is zero, that is,

$$y^3 = \alpha y.$$

There are two cases. If $\alpha \leq 0$ then this equation only has one solution, $y = 0$. The real part of $p(0)$ is 3. It follows that the change in the argument is approximately zero, the argument goes from slightly more than 0 to slightly less than 0.

The change in the argument over the whole of γ is then approximately 4π and so $p(z)$ has two roots in the left half plane.

If $\alpha > 0$ then we cross the real axis when $y = 0$, $\sqrt{\alpha}$ and $-\sqrt{\alpha}$. So we cross the real line three times. When $y = 0$ the real part is 3 and when $y = \pm\sqrt{\alpha}$ the real part is

$$\alpha^2 - 4\alpha + 3 = (\alpha - 2)^2 - 1.$$

This is negative when $\alpha \in (1, 3)$.

There are three cases. If $\alpha \notin [1, 3]$, that is, if either $\alpha > 3$ or $\alpha < 1$ then we only cross the real axis in the right half plane and so the change in the argument is approximately zero over γ_1.

Just as before, this means there are two roots in the left half plane.

If $\alpha \in (1, 3)$ then we first cross the real axis in the left half plane. We started in the first quadrant, so we must be going from the second to the third quadrant. Next we cross the real axis in the right half plane, so we must be going from the fourth quadrant to the first quadrant. Finally we must repeat this whole process, since we start in the first quadrant and we cross the real axis in the left half plane, again. So the change in the argument is approximately 4π.

In total, the change in the argument is approximately 8π, so that it is exactly 8π and the number of zeroes in the left half plane is 4.
It remains to consider the possibility that either $\alpha = 1$ or $\alpha = 3$. If $\alpha = 1$ then $\pm i$ are two roots of $p(z)$. If $\alpha = 3$ then $\pm\sqrt{3}i$ are roots of $p(z)$. In both cases there is almost no change in the argument of $p(z)$ over γ_1 and so the total change in the argument is again 2π and there are two zeroes in the left half plane.
5. Let

$$p(z) = 2z^5 + 6z - 1.$$

Consider $p(x) = 2x^5 + 6x - 1$. We have

$$p(0) = -1 < 0 \qquad \text{and} \qquad p(1) = 2 + 6 - 1 = 7 > 0,$$

and so $p(x)$ has at least one root in the interval $(0, 1)$ by the intermediate value theorem.
We now use Rouchés Theorem to see how many roots $p(z)$ has in the unit disk. Let $f(z) = 6z$ and let $h(z) = 2z^5 - 1$. On the unit circle we have

$$\begin{aligned} |h(z)| &= |2z^5 - 1| \\ &\leq |2z^5| + 1 \\ &= 3 \\ &< 6 \\ &= |6z| \\ &= |f(z)|. \end{aligned}$$

As $f(z)$ has one zero in the unit disk, it follows that $p(z)$ also has one zero. But then $p(z)$ must have one zero on the interval $(0, 1)$.
Now consider how many zeroes $p(z)$ has in the open disk U of radius 2. Let $f(z) = 2z^5$ and let $h(z) = 6z + 1$. On the unit circle we have

$$\begin{aligned} |h(z)| &= |6z + 1| \\ &\leq |6z| + 1 \\ &= 13 \\ &< 64 \\ &= |2z^5| \\ &= |f(z)|. \end{aligned}$$

As $f(z)$ has five zeroes in U, it follows that $p(z)$ also has five zeroes. As it has only one zero in the unit disk it follows that it has four zeroes in the annulus

$$\{\, z \in \mathbb{C} \,|\, 1 < |z| < 2 \,\}.$$

6. We use Rouchés Theorem. Let

$$f(z) = 3z^n \qquad \text{so that} \qquad h(z) = 1 + z + \frac{z^2}{2!} + \dots + \frac{z^m}{m!}.$$

On the unit circle we have

$$\begin{aligned}
|h(z)| &= \left|1 + z + \frac{z^2}{2!} + \dots + \frac{z^m}{m!}\right| \\
&\leq 1 + |z| + \left|\frac{z^2}{2!}\right| + \dots + \left|\frac{z^m}{m!}\right| \\
&= 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{m!} \\
&< 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{m!} + \dots \\
&= e \\
&< 3 \\
&= |3z^n| \\
&= |f(z)|.
\end{aligned}$$

As $f(z)$ has n roots in the unit circle it follows that $p(z)$ also has n roots.

7. Let U be the intersection of a circle of radius R centred at the origin and the right half plane. Let

$$\gamma = \gamma_1 + \gamma_2,$$

be the boundary of U, where γ_2 is the semircircle of radius R starting at $-iR$ and ending at iR and γ_1 is the line segment from iR to $-iR$. Let

$$f(z) = (z-1)^n e^z \qquad \text{and} \qquad h(z) = \lambda(z+1)^n.$$

We check that

$$|h(z)| \leq |f(z)|$$

on γ. First consider what happens on γ_2. We have

$$\begin{aligned}
|e^z| &= |e^{x+iy}| \\
&= |e^x| \cdot |e^{iy}| \\
&= |e^x| \\
&\geq 1,
\end{aligned}$$

as $x \geq 0$ in the right half plane.

We have

$$\begin{aligned}|f(z)| &= |(z-1)^n e^z| \\ &= |(z-1)^n| \cdot |e^z| \\ &\geq |(R-1)|^n \\ &> |\lambda|^n |(R+1)|^n \\ &\geq |\lambda(z+1)^n| \\ &= |h(z)|,\end{aligned}$$

if R is sufficiently large, as $|\lambda| < 1$.
Now consider what happens over γ_1. In this case z is purely imaginary so that

$$|e^z| = 1.$$

Note that

$$|z+1| = |z-1|,$$

since the distance of a point on the imaginary axis to -1 is the same as its distance to 1.
We have

$$\begin{aligned}|f(z)| &= |(z-1)^n e^z| \\ &= |(z-1)^n| \cdot |e^z| \\ &= |z-1|^n \\ &= |z+1|^n \\ &> \lambda^n |z-1|^n \\ &= |\lambda^n (z+1)^n| \\ &= |h(z)|.\end{aligned}$$

Note $f(z)$ has n zeroes in U, since 1 is a zero of order n.
Thus Rouchés theorem implies

$$p(z) = f(z) + h(z) = (z-1)^n e^z + \lambda(z+1)^n$$

has n zeroes in U, which are all in the right hand plane, $\mathrm{Re}(z) > 0$.
Suppose that $\lambda \neq 0$. The derivative of $p(z)$ is

$$q(z) = n(z-1)^{n-1} e^z + (z-1)^n e^z + n\lambda(z+1)^{n-1}$$

A zero that is not a simple zero corresponds to a zero of $p(z)$ that is also a zero of $q(z)$. As $p(z) = 0$, we have

$$(z-1)^n e^z = -\lambda(z+1)^n.$$

Thus

$$q(z) = -n\lambda(z+1)^n - \lambda(z-1)(z+1)^n + n\lambda(z-1)(z+1)^{n-1}.$$

As $\lambda \neq 0$ we have

$$n(z+1)^n + (z-1)(z+1)^n - n(z-1)(z+1)^{n-1} = 0.$$

As we are in the right half plane, $z \neq -1$ and so we deduce

$$n(z+1) + (z-1)(z+1) - n(z-1) = 0.$$

Simplifying we get

$$z^2 + 2n - 1 = 0.$$

If $n = 0$ there is nothing to prove and if $n > 0$ then the solutions to this equation are purely imaginary.
Thus all n zeroes in the right half plane are simple.

Challenge Problems: (Just for fun)

8. Let U be a bounded domain and let $f(z)$ and $h(z)$ be two meromorphic functions on U that are holomorphic on ∂U. Suppose that

$$|h(z)| < |f(z)|$$

on ∂U.
(i) Give an example where $f(z)$ and $f(z)+h(z)$ have a different number of zeroes on U.
(ii) What comparison can we make between $f(z)$ and $f(z) + h(z)$? Prove your assertion.