MODEL ANSWERS TO THE NINTH HOMEWORK

1. The map

$$\alpha\colon \mathbb{C} \longrightarrow \mathbb{C} \qquad \text{given by} \qquad \alpha(z) = \frac{z-a}{R}$$

sends the circle of radius R centred at a to the unit circle. The inverse map is

$$\beta\colon \mathbb{C} \longrightarrow \mathbb{C} \qquad \text{given by} \qquad \beta(z) = Rz + a.$$

As reflection in the unit circle is given by

$$z \longrightarrow \frac{z}{|z|^2}$$

it follows that reflection in the circle

$$\{\, z \in \mathbb{C} \,|\, |z-a| = R \,\}$$

is given by $\beta \circ (z \longrightarrow \frac{z}{|z|^2}) \circ \alpha$. If we apply this to z we get

$$\begin{aligned}
\beta \circ (z \longrightarrow \frac{z}{|z|^2}) \circ \alpha(z) &= \beta \circ (z \longrightarrow \frac{z}{|z|^2}) \left(\frac{z-a}{R}\right) \\
&= \beta\left(\frac{\frac{z-a}{R}}{\frac{|z-a|^2}{R^2}}\right) \\
&= \beta\left(R\frac{z-a}{|z-a|^2}\right) \\
&= R^2\frac{z-a}{|z-a|^2} + a.
\end{aligned}$$

2. Note that reflection in a circle is the composition of a Möbius transformation and complex conjugation. The result is then clear, as complex conjugation maps lines to lines and circles to circles, and a Möbius transformation maps lines and circles to lines and circles.

3. Suppose that $f(z)$ has constant modulus on the circle

$$\{\, z \in \mathbb{C} \,|\, |z-a| = R \,\}.$$

Consider the map

$$\alpha\colon \mathbb{C} \longrightarrow \mathbb{C} \qquad \text{given by} \qquad \alpha(z) = z + a.$$

Let $g = f \circ \alpha$. Then g has constant modulus on the circle of radius R centred at 0.

Suppose that $g(z)$ has a zero of order n at 0 and consider

$$h(z) = \frac{g(z)}{z^n}.$$

Then $h(z)$ is a meromorphic function on the whole plane. The only possible pole of h is at zero but since $g(z)$ has a zero of order n at zero in fact $h(z)$ is an entire function. Note that $h(z)$ also has constant modulus ρ on the circle of radius R and $h(z)$ is non-zero at zero.
By the Schwarz reflection principle

$$h\left(\frac{R^2 z}{|z|^2}\right) = \frac{\rho^2 h(z)}{|h(z)|^2}.$$

It follows that

$$h\left(\frac{1}{z}\right) = \frac{c^2 h(R\bar{z})}{|h(R\bar{z})|^2}.$$

It follows that h is bounded at infinity. Thus $h(z) = c$ is constant by Liouville's theorem. It follows that

$$g(z) = cz^n,$$

and so

$$f(z) = c(z-a)^n.$$

4. As $f(z)$ is bounded as it approaches the unit circle, it follows that the poles of $f(z)$ don't accumulate anywhere. Thus $f(z)$ has finitely many poles. Suppose that the poles are $a_1, a_2, \dots, a_n$, repeated according to multiplicity. Let

$$B(z) = \left(\frac{z-a_1}{1-\bar{a}_1 z}\right)\left(\frac{z-a_2}{1-\bar{a}_2 z}\right)\cdots\left(\frac{z-a_n}{1-\bar{a}_n z}\right).$$

Let $g(z) = B(z)f(z)$. Then $g(z)$ has no poles, since the only possible poles of $g(z)$ are located at the poles of $f(z)$ and the zeroes of the Blaschke product cancel with the poles of $f(z)$. We have

$$\begin{aligned}
\lim_{|z|\to 1} |g(z)| &= \lim_{|z|\to 1} |B(z)f(z)| \\
&= \lim_{|z|\to 1} |B(z)| \cdot \lim_{|z|\to 1} |f(z)| \\
&= 1 \cdot 1 \\
&= 1.
\end{aligned}$$

By the Schwarz reflection principle $g(z)$ extends to a holomorphic function on the whole complex plane and

$$g\left(\frac{1}{\bar{z}}\right) = \frac{1}{\overline{g(z)}}.$$

In particular $g(z)$ extends to a continuous function on the closed unit disk. But then Question 4 on Homework 6 implies that $g(z)$ is a finite Blaschke product

$$C(z) = \left(\frac{z-b_1}{1-\bar{b}_1 z}\right)\left(\frac{z-b_2}{1-\bar{b}_2 z}\right)\cdots\left(\frac{z-b_m}{1-\bar{b}_m z}\right).$$

Dividing by $B(z)$ we see that

$$f(z) = \frac{C(z)}{B(z)}$$

is the quotient of two finite Blaschke products. In particular $f(z)$ is a rational function.

5. The **modulus of an annulus**

$$U = \{ z \in \mathbb{C} \mid a < |z-\alpha| < b \}$$

is defined to be

$$\frac{1}{2\pi}\ln\left(\frac{b}{a}\right).$$

(a) Suppose that the two annuli are

$$U = \{ z \in \mathbb{C} \mid a < |z| < b \} \qquad \text{and} \qquad V = \{ z \in \mathbb{C} \mid c < |z| < d \}.$$

If $f\colon U \longrightarrow V$ is a biholomorphic map then f maps the boundary of U to the boundary of V. Let $g\colon V \longrightarrow U$ be the inverse of f. Therefore if $|z|$ approaches a then $|f(z)|$ approaches a constant (either c or d) and similarly if $|z|$ approaches a then $|f(z)|$ approaches a constant (either d or c). It follows that we can apply the Schwarz reflection principle to both circles $|z| = a$ and $|z| = b$.

The Schwarz reflection principle then says that we can extend f by reflection across either circle. The trick is to keep applying the reflection principle. Applying the reflection principle once to the circle of radius a we can we can exend f to the annulus

$$U_1 = \{ z \in \mathbb{C} \mid a/b < |z| < b \}$$

Note that the boundary circle of radius a/b of U_1 corresponds to the boundary circle of radius b of U. Therefore the modulus of f approaches a constant as we the modulus of z approaches a/b. Applying the reflection principle again, but now to the circle of radius a/b we can extend f to

$$U_2 = \{ z \in \mathbb{C} \mid \frac{a}{b^2} < |z| < b \}.$$

After n iterations of this process we can extend f to the annulus

$$U_n = \{ z \in \mathbb{C} \mid \frac{a}{b^n} < |z| < b \}.$$

If follows that we can extend f to the union

$$\bigcup_{n=1}^{\infty} U_n = \{\, z \in \mathbb{C} \,|\, 0 < |z| < b \,\}.$$

If we reflect this region in the circle of radius b we get the whole punctured plane. Thus f extends to a holomorphic map of the punctured plane. By the same reasoning g extends to a holomorphic map of the punctured complex plane. Consider the composition $g \circ f$. This is the identity map on U, so that $f(g(z)) - z$ is zero on U. It follows that $f(g(z)) - z = 0$ on the whole punctured complex plane. But then $f \circ g$ is the identity. It follows that f extends to a biholomorphic map of the extended complex plane.

(b) Note that the annulus

$$\{\, z \in \mathbb{C} \,|\, a < |z - \alpha| < b \,\} \qquad \text{is biholomorphic to} \qquad \{\, z \in \mathbb{C} \,|\, a < |z| < b \,\},$$

and the annulus

$$\{\, z \in \mathbb{C} \,|\, c < |z - \gamma| < d \,\} \qquad \text{is biholomorphic to} \qquad \{\, z \in \mathbb{C} \,|\, c < |z| < d \,\}.$$

Therefore we may assume that both annuli are centred at the origin.

If

$$\frac{1}{2\pi} \ln\left(\frac{b}{a}\right) = \frac{1}{2\pi} \ln\left(\frac{d}{c}\right).$$

then multiplying by 2π and exponentiating we see that

$$\frac{b}{a} = \frac{d}{c}.$$

Consider the biholomorphic map

$$z \longrightarrow \frac{dz}{b}$$

The sends the circle of radius b to the circle of radius d. It also sends the circle of radius a to the circle of radius

$$\begin{aligned} \frac{da}{b} &= \left(\frac{bc}{a}\right) \frac{a}{b} \\ &= c. \end{aligned}$$

Thus the two annuli are biholomorphic.

Now suppose there is a biholomorphic map between the two annuli. By part (a) there is a biholomorphic map f of the punctured complex plane which sends U to V. By Question 7 of Homework 7, f is a Möbius transformation which permutes 0 and ∞. The map

$$\alpha \colon \mathbb{C} \longrightarrow \mathbb{C} \qquad \text{given by} \qquad z \longrightarrow \frac{cd}{z}$$

switches 0 and ∞ and sends V to V. Possibly replacing f by $\alpha \circ f$ we may assume that f fixes 0 and ∞. As f fixes ∞ it follows that

$$f(z) = \lambda z + \mu \qquad \text{where} \qquad \lambda, \mu \in \mathbb{C}$$

and as f fixes 0, $\mu = 0$. As the circle of radius a is sent to the circle of radius c

$$|\lambda| = \frac{b}{a}$$

and as the circle of radius b is sent to the circle of radius d we have

$$|\lambda| = \frac{d}{c}.$$

But then

$$\frac{b}{a} = \frac{d}{c}$$

so that the two moduli are the same.
(c) Every such map is given by a Möbius transformation f that permutes 0 and ∞. The involution

$$\alpha \colon \mathbb{C} \longrightarrow \mathbb{C} \qquad \text{given by} \qquad z \longrightarrow \frac{ab}{z}$$

switches 0 and ∞ and sends U to U. Thus possibly composing with α we may assume that

$$f(z) = \lambda z,$$

where $|\lambda| = 1$. But then f is a rotation.
6. Let

$$U = \{\, z \in \Delta \mid \operatorname{Re}(z) > 0 \,\}$$

be the right half of the unit disk and let

$$g \colon U \longrightarrow \Delta$$

be *the* biholomorphic map that fixes the points $\pm i$ and 1.
(a) h is a composition of an angle reversing map, a conformal map and an angle reversing map. It follows that h is also conformal. As it is a bijection it follows that h is biholomorphic. We have

$$\begin{aligned} h(1) &= \overline{g(\bar{1})} \\ &= \overline{g(1)} \\ &= \bar{1} \\ &= 1, \end{aligned}$$

and

$$\begin{aligned} h(\pm i) &= \overline{g(\overline{\pm i})} \\ &= \overline{g(\mp i)} \\ &= \overline{\mp i} \\ &= \pm i. \end{aligned}$$

Thus h is a biholomorphic map that also fixes the points $\pm i$ and 1.
(b) The right half circle is biholomorphic with the unit circle, which is in turn biholomorphic to the upper plane. Any biholomorphic map of the upper half plane which fixes three points of the boundary is the identity. Therefore the same is true of the right half plane. If f is the inverse of g then $f \circ h$ fixes three points of the boundary. Therefore $f \circ h$ is the identity. It follows that f is the inverse of h so that $g = h$. But then

$$\begin{aligned} \overline{g(z)} &= h(\bar{z}) \\ &= g(\bar{z}). \end{aligned}$$

(c) Note that if z is real then $g(z)$ is real, using (b). On the other hand, $g(0)$ is a point of the unit circle. It cannot be 1 and so $g(0) = -1$.
(d) We first map U to the first quadrant. The map

$$z \longrightarrow i\frac{1-z}{1+z}$$

introduced in Example 2 of Lecture 23 maps the upper semicircle to the first quadrant. The upper half of the unit circle maps to the non-negative reals and the real line gets mapped to the non-negative imaginary numbers. If we precompose with rotation through $\pi/2$

$$z \longrightarrow iz$$

we get a biholomorphic map that sends U to the first quadrant. If we square

$$z \longrightarrow z^2$$

this sends the first quadrant to the upper half plane. Finally the map

$$z \longrightarrow \frac{z-i}{z+i}$$

sends the upper half plane to the unit circle. If we again rotate by $\pi/2$ then we get the map $g(z)$ we are looking for:

$$\begin{aligned} 1 \longrightarrow i \longrightarrow 1 \longrightarrow 1 \longrightarrow -i \longrightarrow 1 \\ i \longrightarrow -1 \longrightarrow \infty \longrightarrow \infty \longrightarrow 1 \longrightarrow i \\ -i \longrightarrow 1 \longrightarrow 0 \longrightarrow 0 \longrightarrow -1 \longrightarrow -i. \end{aligned}$$

Now g sends 0 to

$$0 \longrightarrow 0 \longrightarrow i \longrightarrow -1 \longrightarrow i \longrightarrow -1.$$

7. If $f\colon \mathbb{C} \longrightarrow \Delta$ is a holomorphic map then f is constant by Liouville's theorem. In particular Δ and $\mathbb{C}$ are not biholomorphic.
If $g\colon \mathbb{P}^1 \longrightarrow \Delta$ is a holomorphic map then we get a holomorphic map $f\colon \mathbb{C} \longrightarrow \Delta$ by restriction. As we already decided this map is constant it follows that g is not biholomorphic. Thus Δ and $\mathbb{P}^1$ are not biholomorphic.
If $g\colon \mathbb{P}^1 \longrightarrow \mathbb{C}$ is holomorphic then consider the restriction $f\colon \mathbb{C} \longrightarrow \mathbb{C}$. This is a holomorphic map which is bounded at infinity. Thus f is constant by Liouville's theorem. Thus g is not biholomorphic and so $\mathbb{C}$ and $\mathbb{P}^1$ are not biholomorphic.
For those who know a little topology, note that $\mathbb{P}^1$ is homeomorphic to the sphere S^2, which is compact, as it is a closed and bounded subset of $\mathbb{R}^3$. As Δ and $\mathbb{C}$ are not compact, it follows that Δ and $\mathbb{C}$ are not even homeomorphic to $\mathbb{P}^1$, let alone biholomorphic.
8. First observe that f induces a biholomorphic map of U_r and the open disk of radius r centred around the origin. Let

$$\alpha\colon \mathbb{C} \longrightarrow \mathbb{C}$$

be the map given by

$$z \longrightarrow \frac{z}{r}$$

This induces a biholomorphic map of the open disk of radius r and the unit disk. The composition

$$\alpha \circ f\colon U_r \longrightarrow \Delta$$

is a biholomorphic map.

Challenge Problems: (Just for fun)

9. Consider reflection $z \longrightarrow z^*$ across the parabola $y = x^2$.
(a) Find an expression for z^*.
(b) Expand z^* in a power series in powers of $\bar{z}$.
(c) Determine the radius of convergence of the power series from (b).
(d) Try to explain graphically why the radius of convergence is finite.
10. The words "AMBULANCE" are written backwards on the front of an ambulance so that when a driver sees an ambulance approaching from behind in their rear view mirror the words appear normally. Why aren't the letters upside down as well?