# HOMEWORK 0, DUE TWELTH OF NEVER

1. Identify and sketch the set of points satisfying:

(a)
$$|z - 1 - i| = 1.$$
(b)
$$1 < |2z - 6| < 2.$$
(c)
$$|z - 1|^2 + |z + 1|^2 < 8.$$
(d)
$$|z - 1| + |z + 1| < 2.$$
(e)
$$|z - 1| < |z|.$$
(f)
$$0 < \operatorname{Im} z < \pi.$$
(g)
$$-\pi < \operatorname{Re} z < \pi.$$
(h)
$$|\operatorname{Re} z| < |z|.$$
(i)
$$\operatorname{Re}(iz + 2) > 0.$$
(j)
$$|z - i|^2 + |z + i|^2 < 2.$$

2. Verify the following identities, from the definitions:

(a)
$$\overline{z + w} = \bar{z} + \bar{w}.$$
(b)
$$\overline{zw} = \bar{z}\bar{w}.$$
(c)
$$|\bar{z}| = |z|.$$
(d)
$$|\bar{z}|^2 = z\bar{z}.$$

Draw sketches to illustrate (a) and (c).

3. Fix a real number $\rho > 0$, $\rho \neq 1$ and fix $z_0$ and $z_1 \in \mathbb{C}$. Show that the set
$$|z - z_0| = \rho|z - z_1|$$

is a circle. Sketch this circle for $\rho = 1/2$ and 2 when $z_0 = 0$ and $z_1 = 1$. What happens when $\rho = 1$?

**Challenge Problems:** (Just for fun)

4. Consider functions

$$f = f_0 \colon \mathbb{R} \longrightarrow \mathbb{R}$$

given by the rule

$$f(x) = \begin{cases} \frac{p(x)}{x^m} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0, \end{cases}$$

where $p(x)$ is a polynomial and $m \geq 0$ is an integer. We are going to show that all of these functions are infinitely differentiable but that all of them have a MacLaurin series which is zero.

(i) Show that $f(x)$ is continuous.

(ii) Show that $f(x)$ is differentiable and that the derivative at the origin is zero.

(iii) Show that there is a polynomial $q(x)$ and an integer $n \geq 0$ such that

$$f_1(x) = \begin{cases} \frac{q(x)}{x^n} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0, \end{cases}$$

where $f_1(x)$ is the derivative of $f(x)$.

(iv) Conclude that $f(x)$ is infinitely differentiable and that all of its derivatives are zero at the origin.

(v) Show that the MacLaurin series (that is, the Taylor series centred at the origin) is identically zero.