HOMEWORK 1, DUE FRIDAY JANUARY 17TH, 12PM

1. Let \( a \) be a complex number and let \( \rho > 0 \) be a positive real. Show that the equation

\[
|z|^2 - 2 \text{Re}(\bar{a}z) + |a|^2 = \rho^2
\]

represents a circle centred at \( a \) with radius \( \rho \).

2. Consider the polynomial \( p(z) = z^3 + z^2 + z + 1 \).
   (a) Verify that \( i \) is a root of \( p(z) \).
   (b) Find the other roots.

3. For which integers \( n \) is \( i \) an \( n \)th root of unity?

4. Let \( n \geq 1 \) be an integer.
   (a) Show that

\[
1 + z + z^2 + z^3 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}.
\]

   (b) Show that

\[
1 + \cos \theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin(n + \frac{1}{2})\theta}{2 \sin \frac{\theta}{2}}.
\]

5. Show that

\[
\left( \frac{1 + i \tan \theta}{1 - i \tan \theta} \right)^n = \frac{1 + i \tan n\theta}{1 - i \tan n\theta}
\]

for every integer \( n \).

6. Find the six distinct roots of \(-64\).

7. Show that

\[
(1 = \sqrt{3}i)^{10} = 2^{-11}(-1 + \sqrt{3}i).
\]

**Challenge Problems:** (Just for fun)

8. Prove the triangle inequality

\[
|z + w| \leq |z| + |w|,
\]

with equality if and only if either \( z = 0 \) or \( w \) is a real scalar multiple of \( z \).

9. Given three distinct points \( a, b \) and \( c \) of the extended complex plane (so that \( a, b \) and \( c \) are either complex numbers or \( \infty \)) show that there is a unique Möbius transformation

\[
z \rightarrow \frac{az + b}{cz + d}
\]
taking $a$ to 0, $b$ to 1 and $c$ to $\infty$. 