HOMEWORK 5, DUE FRIDAY FEBRUARY 14TH, 12PM

1. Show that the following functions are holomorphic and calculate their derivatives $f'(z)$:

(a)

$$f(z) = (2z^2 + 1)^5$$

(b)

$$f(z) = \frac{z-1}{2z+1} \qquad \text{on the region} \qquad z \neq -\frac{1}{2}$$

(c)

$$f(z) = \frac{(1+z^2)^4}{z^2} \qquad \text{on the region} \qquad z \neq 0$$

(d)

$$f(z) = \tan z = \frac{\sin z}{\cos z} \qquad \text{on the region} \qquad \cos z \neq 0.$$

(e)

$$f(z) = \cosh(z)$$

(f)

$$f(z) = \sinh(z)$$

2. Show that $f(z) = \bar{z}$ is not holomorphic.
(a) Using the definition.
(b) Using the Cauchy-Riemann equations.
3. Let $f(z) = u(z) + iv(z)$ be a holomorphic function, where u and v are real valued. If we put $z = re^{i\theta}$ into polar coordinates then

$$u(r, \theta) = u(re^{i\theta}) \qquad \text{and} \qquad v(r, \theta) = v(re^{i\theta})$$

become functions of r and θ.
(a) Derive the polar form of the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \text{and} \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}.$$

(b) Check that the functions

$$u(r, \theta) = r^m \cos(m\theta) \qquad \text{and} \qquad v(r, \theta) = r^m \sin(m\theta)$$

satisfy the polar form of the Cauchy-Riemann equations, where m is an integer.

4. Let f be a bounded holomorphic function on a bounded region U. If f is one to one (or injective) then show that the area of the image $f(U)$ of U is

$$\iint_U |f'(z)|^2 \, \mathrm{d}x\mathrm{d}y.$$

Challenge Problems: (Just for fun)

5. We are going to go through Euler's derivation of the value of $\zeta(2)$.
(a) What are the roots of

$$\sin(\pi z) = 0?$$

(b) If $p(z)$ is a polynomial with roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ it follows that

$$p(z) = \alpha \left(1 - \frac{z}{\alpha_1}\right)\left(1 - \frac{z}{\alpha_2}\right) \ldots \left(1 - \frac{z}{\alpha_n}\right),$$

where α is a complex number (we divide through by the product of the roots for convergence of the infinite product below). Assuming the same sort of result holds for $\sin(\pi z)$, find an infinite product expansion for $\sin(\pi z)$.
(c) Grouping positive and negative roots together, conclude that

$$\sin(\pi z) = \pi z \left(1 - \frac{z^2}{1}\right)\left(1 - \frac{z^2}{4}\right)\left(1 - \frac{z^2}{9}\right), \ldots.$$

(d) Multiply out the RHS of (c) to get the power series for $\sin(\pi z)$.
(e) Compare coefficients of z^3 to get a formula for $\zeta(2)$.