HOMEWORK 6, DUE FRIDAY FEBRUARY 21ST, 12PM

1. Let $f\colon U \longrightarrow \mathbb{C}$ be a holomorphic function. Show that f is constant if
(a) the real part of f is constant.
(b) the imaginary part of f is constant.
You may use the fact that if u is a real valued function on U such that

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0,$$

on U then u is constant.
2. Let

$$h\colon [0, 1] \longrightarrow \mathbb{C}$$

be a continuous complex valued function defined on the unit interval $[0, 1]$. Define a function

$$f\colon U \longrightarrow \mathbb{C} \qquad \text{by the rule} \qquad f(z) = \int_0^1 \frac{h(t)}{t - z}\, \mathrm{d}t,$$

where U is the region $\mathbb{C} \setminus [0, 1]$.
Show that f is holomorphic on U by using the limit definition of the derivative. What is $f'(z)$?
3. Sketch the families of level curves for the following functions $f = u + iv$.
(a)

$$f(z) = \frac{1}{z}.$$

(b)

$$f(z) = \frac{1}{z^2}.$$

(c)

$$f(z) = z^6.$$

Determine where $f(z)$ is conformal and where it is not conformal.
4. Suppose that P and Q are two functions on the annulus

$$V = \{\, z \in \mathbb{C} \mid a < |z| < b \,\}$$

which have continuous partial derivatives. If

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

then show that the integral

$$\int_{\gamma_r} P\,\mathrm{d}x + Q\,\mathrm{d}y$$

is independent of r, where γ_r is the circle of radius $r \in (a, b)$ centred at the origin and we traverse γ_r counterclockwise.

5. Compute

$$\int_\gamma \frac{z+2}{z}\,\mathrm{d}z$$

when

(a) γ is the semicircle of radius 2 centred at the origin in the upper half plane.

(b) γ is the semicircle of radius 2 centred at the origin in the lower half plane.

(c) γ is the circle of radius 2 centred at the origin.

All arcs are traversed counterclockwise.

6. Show that if U is a bounded region with smooth boundary then the area of U is given by the integral

$$\frac{1}{2i}\int_{\partial U} \bar{z}\,\mathrm{d}z.$$

Challenge Problems: (Just for fun)

1. (continued)

(c) the modulus of f is constant.

(d) the argument of f is constant.

7. (Circles of Appolonius) Fix two points a and b in $\mathbb{C}$.

(a) Fix a positive real l. Show that loci of points z such that the ratio between the distance to a and the distance to b is l is a circle C_1.

(b) Let C_2 be a circle containing a and b. Show that C_1 and C_2 are orthogonal circles.

(*Hint: Use Möbius transfermations.*)