HOMEWORK 8, DUE FRIDAY MARCH 6TH, 12PM

1. Let $f$ be an entire function. If the real part of $f$ is bounded from above then show that $f$ is constant. (*Hint: consider $e^{f(z)}$.*

2. Suppose that $f$ is an entire function such that $f(z)/z^n$ is bounded for $|z| \geq R$. Show that $f$ is a polynomial of degree at most $n$.

3. Expand the following functions in power series about $\infty$:
   (a) $\frac{1}{z^2 + 1}$;
   (b) $\frac{z^2}{z^3 - 1}$;
   (c) $e^{1/z^2}$;
   (d) $z \sinh(1/z)$.

4. Let $E$ be a closed bounded subset of the complex plane $\mathbb{C}$ over which area can be defined and set
   $$ f(w) = \iint_E \frac{dx\,dy}{w - z} \quad \text{where} \quad w \in U = \mathbb{C} \setminus E $$
   and $z = x + iy$. Show that $f$ is holomorphic at $\infty$ and find a formula for the coefficients.

5. Find the zeroes and their orders of the following functions:
   (a) $\frac{z^2 + 1}{z^2 - 1}$.
   (b) $\frac{1}{z} + \frac{1}{z^5}$.
   (c) $z^2 \sin z$.

**Challenge Problems:** (Just for fun)

6. Consider the power series
   $$ \sum_n z^n! = z + z^2 + z^6 + z^{24} + z^{120} + \ldots $$
Show that the radius of convergence is one so that

\[ f(z) = \sum_{n} z^n! \]

is a holomorphic function on the open unit disk. On the other hand show that

\[ \lim_{r \to 1} |f(r\omega)| = \infty \]

where \( \omega \) is any root of unity and \( r \) approaches 1 from below.

7. If \( f \) is an entire function and \( f \) omits all of the values in a non-empty open disk then show that \( f \) is constant.