16. Liouville’s Theorem

To apply Cauchy’s formula we will need some easy estimates.

Definition 16.1. Let

\[ \gamma : [\alpha, \beta] \longrightarrow \mathbb{C}, \]

be a differentiable curve. The length of \( \gamma \) is the integral

\[ L = \int_{\alpha}^{\beta} (x'(t)^2 + y'(t))^{1/2} \, dt. \]

If one picks points \( a = z_0, z_1, \ldots, z_n = b \), where \( a = \gamma(\alpha) \) and \( b = \gamma(\beta) \) then the distance from \( z_i \) to \( z_{i+1} \) along \( \gamma \) is approximated by the length of the line connecting \( z_i \) to \( z_{i+1} \), which is

\[ \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2} \]

by Pythagoras. By the mean value theorem we can find \( \tau_i \) and \( \upsilon_i \) in the interval \( (t_i, t_{i+1}) \) such that

\[ x_{i+1} - x_i = x'(\tau_i)(t_{i+1} - t_i) \quad \text{and} \quad y_{i+1} - y_i = y'(\upsilon_i)(t_{i+1} - t_i). \]

Thus the length of the line connecting \( z_i \) to \( z_{i+1} \) is

\[ \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2} = \sqrt{(x'(\tau_i)(t_{i+1} - t_i))^2 + (y'(\upsilon_i)(t_{i+1} - t_i))^2} \]

Summing over \( i \) we get a Riemann sum approximating the integral in (16.1).

Note that

\[ (x'(t)^2 + y'(t))^{1/2} = |\gamma'(t)| \]

is the length of the tangent vector \( \gamma \) at \( t \). Thus the length is also

\[ L = \int_{\alpha}^{\beta} |\gamma'(t)| \, dt, \]

the integral of the speed.

If \( \gamma \) is piecewise differentiable, we can define the length by simply adding together the lengths of the differentiable pieces.

If

\[ \gamma : [\alpha, \beta] \longrightarrow U \]

is a curve and \( f : U \longrightarrow \mathbb{C} \) is continuous then \( M \) denotes the maximum value of the absolute value of \( f \) over the curve \( \gamma \):

\[ M = \sup_{t \in [\alpha, \beta]} |f(\gamma(t))|. \]

We have the following basic but very useful:
Lemma 16.2. Let \( f: U \rightarrow \mathbb{C} \) be a continuous function over a region \( U \) and let 
\[
\gamma: [\alpha, \beta] \rightarrow U,
\]
be a piecewise differentiable curve. Then
\[
\left| \int_\gamma f(z) \, dz \right| \leq LM.
\]
It is easy to check (16.2) by using Riemann sums and the triangle inequality.

Theorem 16.3 (Liouville’s theorem). Every bounded entire function is constant.

Proof. By assumption there is a real number \( M_0 \) such that 
\[
|f(z)| \leq M_0.
\]
As \( f(z) \) is entire it has a power series expansion whose radius of convergence is \( \infty \), 
\[
f(z) = \sum_n a_n z^n.
\]
The coefficients are given by Cauchy’s formula 
\[
a_n = \frac{1}{2\pi i} \oint_{|z|=r} \frac{f(z) \, dz}{z^{n+1}},
\]
where the radius is any positive real number \( r \). We estimate the absolute value of \( a_n \).
The circle of radius \( r \) centred at the origin has length 
\[
L = 2\pi r.
\]
We also have 
\[
\frac{|f(z)|}{z^{n+1}} = \frac{|f(z)|}{r^{n+1}} \leq \frac{M_0}{r^{n+1}}.
\]
implies that

\[ |a_n| = \left| \frac{1}{2\pi i} \oint_{|z|=r} \frac{f(z) \, dz}{z^{n+1}} \right| \leq LM \leq 2\pi r \frac{M_0}{2\pi r^{n+1}} = M_0 \frac{r^n}{r^n}. \]

As \( r \) tends to infinity the last quantity tends to zero if \( n > 0 \). The only possibility is that \( |a_n| = 0 \) so that \( a_n = 0 \).

Thus \( f(z) = a_0 \) is a constant. \( \square \)

The inequality

\[ |a_n| r^n \leq \sup_{|z|=r} |f(z)| \]

is sometimes known as Cauchy's inequality.

It is convenient to introduce the notion of the limit at \( \infty \). One common trick in real variables is to use the fact that a function \( h(x) \) tends to infinity if and only if \( 1/h(x) \) tends to infinity. We can do the same thing in complex variable but now for the input as well as the output:

**Definition 16.4.** Let \( U \subset \mathbb{C} \) be a region. We say that \( U \) is a neighbourhood of \( \infty \) if there is a real number \( R \) such that if \( |z| > R \) then \( z \in U \).

Let \( f: U \rightarrow \mathbb{C} \) be a function defined on a region \( U \) which is a neighbourhood of infinity. The limit of \( f(z) \) as \( z \) goes to infinity is

\[ \lim_{z \to \infty} f(z) = \lim_{w \to 0} f \left( \frac{1}{w} \right). \]

Note that \( w \) tends towards zero if and only if \( |w| \) tends towards zero if and only if \( |z| \) tends towards \( \infty \). Note also that \( U \) is a neighbourhood of infinity if and only if the image of \( U \) under the reciprocal map contains an open disk centred at the origin.

**Theorem 16.5** (Fundamental theorem of algebra). If \( p(z) \) is a complex polynomial of degree \( n > 0 \) then \( p(z) \) has a complex root, that is, there is a complex number \( \alpha \) such that

\[ p(\alpha) = 0. \]
Proof. Suppose that $p(z)$ is a polynomial with no roots. We are going to show that $p(z)$ has degree zero.

Let

$$f(z) = \frac{1}{p(z)}.$$ 

As we are assuming that $p(z)$ is never zero, it follows that $f(z)$ is entire.

Suppose that

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0,$$

where $a_n \neq 0$. There is no harm in dividing through by $a_n$ so that $a_n = 1$. Consider

$$\frac{p(z)}{z^n} = 1 + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots + \frac{a_0}{z^n}.$$ 

As $z$ goes to infinity this tends to 1. Thus $|p(z)|$ is bounded away from zero and so $|f(z)|$ is bounded from above. But then $f$ is constant by Liouville’s theorem so that $p(z)$ is constant. It follows that the degree of $p(z)$ is zero. \qed