20. **Isolated Singularities**

**Definition 20.1.** Let \( f : U \to \mathbb{C} \) be a holomorphic function on a region \( U \). Let \( a \notin U \).

We say that \( a \) is an **isolated singularity** of \( f \) if \( U \) contains a punctured neighbourhood of \( a \).

Note that \( \log z \) does not have an isolated singularity at 0, since we have to remove all of \((-\infty, 0]\) to get a continuous function. By contrast its derivative \( 1/z \) is holomorphic except at 0 and so it has an isolated singularity at 0.

Suppose that \( f \) has an isolated singularity at \( a \). As a punctured neighbourhood of \( a \) is a special type of annulus, \( f \) has a Laurent expansion centred at \( a \),

\[
f(z) = \sum_{k=-\infty}^{\infty} a_k (z - a)^k,
\]

valid for

\[0 < |z - a| < r,\]

for some real \( r \).

The behaviour at \( a \) is dictated by the negative part of the Laurent expansion.

**Definition 20.2.** If \( f \) has an isolated singularity at \( a \) and all of the coefficients \( a_k \) of the Laurent expansion

\[
f(z) = \sum_{k=-\infty}^{\infty} a_k (z - a)^k,
\]

vanish if \( k < 0 \), then we say that \( f \) has a **removable singularity**.

If \( f \) has a removable singularity then in fact we can extend \( f \) to a holomorphic function in a neighbourhood of \( a \). Indeed, the Laurent expansion of \( f \) is a power series expansion, and this defines a holomorphic function in a neighbourhood of \( a \).

**Example 20.3.** The function

\[
\frac{\sin z}{z}
\]

has a removable singularity at \( a \).
Indeed,
\[
\frac{\sin z}{z} = \frac{1}{z} \left( z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots \right)
= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \ldots,
\]
is the Laurent series expansion of \( \frac{\sin z}{z} \). Visibly there are no negative terms, so visibly
\[
\frac{\sin z}{z}
\]
extends to a holomorphic function.

**Theorem 20.4** (Riemann’s theorem on removable singularities). Let \( f(z) \) be a holomorphic function which has an isolated singularity at \( a \).

Then \( f(z) \) has a removable singularity at \( a \) if and only if \( f(z) \) is bounded near \( a \).

**Proof.** One direction is clear. If \( f(z) \) is holomorphic at \( a \) then it is bounded at \( a \).

Now suppose that \( f(z) \) is bounded near \( a \). Consider the Laurent expansion of \( f \) centred at \( a \):
\[
f(z) = \sum_k a_k (z - a)^k.
\]

Note that
\[
a_k = \frac{1}{2\pi i} \oint_{|z-a|=r} \frac{f(z)}{(z-a)^{k+1}} \, dz,
\]
for any sufficiently small circle of radius \( r \) centred at \( a \). We have
\[
|a_k| \leq LM,
\]
where \( L \) is the length of the circle and \( M \) is the largest value of the absolute value of \( f(z) \).

The length \( L \) of the circle is \( 2\pi r \). By hypothesis there is a constant \( M_0 \) such that
\[
|f(z)| \leq M_0,
\]
near \( a \). Thus
\[
\left| \frac{f(z)}{z^{n+1}} \right| = \left| \frac{f(z)}{|z^{n+1}|} \right|
= \frac{|f(z)|}{r^{n+1}}
\leq \frac{M_0}{r^{n+1}},
\]
(16.2) implies that

\[ |a_n| = \left| \frac{1}{2\pi i} \oint_{|z|=r} \frac{f(z) \, dz}{z^{n+1}} \right| \leq LM \leq 2\pi r \frac{M_0}{2\pi r^{n+1}} = \frac{M_0}{r^n}. \]

As \( r \) tends to zero the last quantity tends to zero if \( n < 0 \). The only possibility is that

\[ |a_n| = 0 \quad \text{so that} \quad a_n = 0. \]

Thus \( f(z) \) is given by a convergent power series close to \( a \), so that \( f \) extends to a holomorphic function near \( a \). \( \square \)

**Definition 20.5.** If \( f \) has an isolated singularity at \( a \) and all of the coefficients \( a_k \) of the Laurent expansion

\[ f(z) = \sum_{k=-\infty}^{\infty} a_k(z - a)^k, \]

vanish if \( k < -n \) but \( a_{-n} \neq 0 \) then we say that \( f \) has a **pole of order** \( n \) at \( a \).

**Example 20.6.** The function

\[ \frac{\cos z}{z} \]

has a pole of order 1 at 0.

**Theorem 20.7.** Let \( f(z) \) be a holomorphic function with an isolated singularity at \( a \).

The following are equivalent:

1. \( f \) has a pole of order \( n \) at \( a \).
2. there is a function \( g(z) \) holomorphic and non-zero at \( a \) such that

\[ f(z) = \frac{g(z)}{(z - a)^n}. \]

3. The function

\[ \frac{1}{f(z)} \]

is holomorphic at \( a \) and has a zero of order \( n \) at \( a \).
Proof. Suppose that (1) holds, suppose that $f$ has a pole of order $n$. Then the Laurent expansion of $f$ looks like

$$f(z) = \frac{a_{-n}}{(z-a)^n} + \frac{a_{-n+1}}{(z-a)^{n-1}} + \cdots + \frac{a_1}{(z-a)} + a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots.$$ 

Let

$$g(z) = a_{-n} + a_{-n+1}(z-a) + a_{-n+2}(z-a)^2 + \ldots.$$ 

Then $g$ is holomorphic at $a$ and we have

$$f(z) = \frac{g(z)}{(z-a)^n}.$$ 

Note that

$$g(a) = a_{-n} \neq 0.$$ 

Now suppose that (2) holds. Then

$$\frac{1}{f(z)} = \frac{(z-a)^n}{g(z)}.$$ 

As $g(a) \neq 0$ this is holomorphic at $a$.

Now suppose that (3) holds. Then we may write

$$\frac{1}{f(z)} = (z-a)^n g(z),$$

where $g(z)$ is holomorphic and non-zero at $a$. In this case

$$f(z) = \frac{1}{(z-a)^n} h(z)$$

where

$$h(z) = \frac{1}{g(z)}$$

is holomorphic at $a$. As $h(z)$ is holomorphic at $a$, it has a power series expansion

$$h(z) = \sum_{k \geq 0} a_k (z-a)^k.$$ 

As $h(z)$ is the reciprocal of a non-zero function $a_0 \neq 0$. Dividing through by $(z-a)^n$ we get

$$f(z) = \frac{a_0}{(z-a)^n} + \frac{a_1}{(z-a)^{n-1}} + \ldots.$$ 

This is a Laurent series expansion starting in degree $-n$ so that $f$ has a pole of order $n$. 

The final possibility for an isolated singularity is:
Definition 20.8. Let \( f \) be a holomorphic function with an isolated singularity at \( a \).

We say that \( a \) is an essential singularity of \( f(z) \) if the Laurent series expansion has infinitely many non-zero negative terms.

Example 20.9.

\[
\sin \left( \frac{1}{z} \right)
\]

has an essential singularity at 0.

Indeed

\[
\sin \left( \frac{1}{z} \right) = \cdots + \frac{1}{5!z^5} - \frac{1}{3!z^3} + \frac{1}{z}.
\]