4. Roots of Unity

**Theorem 4.1** (De Moivre’s Theorem).

\[(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.\]

**Proof.** We have

\[(\cos \theta + i \sin \theta)^n = (e^{i\theta})^n
= e^{in\theta}
= \cos n\theta + i \sin n\theta.\]

One can use this to derive simple formulas. For example suppose we want to compute triple angle formulas. We use (4.1) to when \(n = 3\). We can expand the LHS using the binomial theorem.

\[(\cos \theta + i \sin \theta)^3 = \cos^3 \theta + 3i \cos^2 \theta \sin \theta - 3 \cos \theta \sin^2 \theta - i \sin^3 \theta.\]

Equating real and imaginary parts we get

\[
\begin{align*}
\cos 3\theta &= \cos^3 \theta - 3 \cos \theta \sin^2 \theta \\
\sin 3\theta &= 3 \cos^2 \theta \sin \theta - \sin^3 \theta.
\end{align*}
\]

We can also use Euler’s formula to compute \(n\)th roots.

**Example 4.2.** What are the cube roots of 125?

We are looking for complex numbers \(z\) such that

\[z^3 = 125.\]

We write \(z\) in polar form

\[z = re^{i\theta}.\]

Then we get the equation

\[r^3 e^{3i\theta} = 125.\]

Taking the modulus of both sides we see that

\[r^3 = 125.\]

As \(r\) is a non-negative real number it follows that

\[r = 5.\]

If we cancel 125 from both sides, we are reduced to solving

\[e^{3i\theta} = 1,\]

that is, we are trying to find all cube roots of 1.
What are the possible arguments of such complex numbers? One possibility is clear, $\theta = 0$. In other words, 1 is a cube root of one. Another possibility is that

$$3\theta = 2\pi,$$

so that when we add $\theta$ to itself we go once around the origin. This gives the solution

$$\theta = \frac{2\pi}{3}.$$

It follows that

$$\omega = \frac{1}{2} \left(-1 + \sqrt{3}i\right)$$

is a cube root of one.

A third possibility is that we go twice around the origin, so that

$$3\theta = 4\pi \quad \text{and} \quad \theta = \frac{4\pi}{3}.$$

In this case we get the last cube root of one

$$\omega' = \frac{1}{2} \left(-1 - i\sqrt{3}\right).$$

Note some interesting connections between the roots. First off $\omega'$ is the complex conjugate of $\omega$:

$$\omega' = \overline{\omega}.$$

In fact it is a general fact that the roots of a real polynomial come in complex conjugate pairs.

**Lemma 4.3.** Let $p(x)$ be a real polynomial.

Then the roots of $p(x)$ come in complex conjugate pairs.

**Proof.** We may suppose that

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0,$$

where $a_0, a_1, \ldots, a_n$ are real numbers. Let $\alpha$ be a root of $p(x)$. We have

$$p(\bar{\alpha}) = a_n (\bar{\alpha})^n + a_{n-1} (\bar{\alpha})^{n-1} + \cdots + a_0$$

$$= a_n \overline{\alpha^n} + a_{n-1} \overline{\alpha^{n-1}} + \cdots + a_0$$

$$= a_n \overline{\alpha^n} + a_n \overline{\alpha^n} + \cdots + a_0$$

$$= a_n \alpha^n + a_{n-1} \alpha^{n-1} + \cdots + a_0$$

$$= p(\alpha)$$

$$= \overline{0}$$

$$= 0.$$

Thus $\bar{\alpha}$ is also a root of $p(x)$.
For example, for the polynomial
\[ z^3 - 1 = (z - 1)(z - \omega)(z - \omega'), \]
if we take the complex conjugate, the LHS is unchanged. On the RHS, 1 is fixed and so complex conjugation must switch \( \omega \) and \( \omega' \).

Secondly note that \( \omega' = \omega^2 \).

This is again a general fact:

**Lemma 4.4.** If \( \zeta \) is an \( n \)th root of unity then so are all powers of \( \zeta \).

**Proof.** Consider \( \alpha = \zeta^i \). We have
\[
\begin{align*}
\alpha^n &= (\zeta^i)^n \\
&= \zeta^{in} \\
&= (\zeta^n)^i \\
&= 1^i \\
&= 1.
\end{align*}
\]