FIRST MIDTERM
MATH 120A, UCSD, WINTER 20

You have 50 minutes.
There are 4 problems, and the total number of points is 50. Show all your work. Please make your work as clear and easy to follow as possible.

Name: __________________________
Signature: ________________________
Student ID #: ____________________
Section instructor: ________________
Section Time: ____________________

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15</td>
<td></td>
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<tr>
<td>2</td>
<td>10</td>
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<td>Total</td>
<td>50</td>
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1. (15pts) (i) *Give the definition of the principal value of the argument.*

If \( z \) is a complex number the principal value of the argument, denoted \( \text{Arg}(z) \), is the angle the vector \((x, y)\) makes with the \(x\)-axis, with values constrained to lie in the range \((-\pi, \pi]\).

(ii) *Give the definition of an open disk.*

If \( a \in \mathbb{C} \) is a complex number and \( \epsilon > 0 \) is a positive real, the open disk centred at \( a \) is the set of complex numbers whose distance to \( a \) is less than \( \epsilon \).

(iii) *Give the definition of a Möbius transformation.*

Any function

\[
M : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \quad \text{of the form} \quad z \rightarrow \frac{az + b}{cz + d},
\]

where \( a, b, c \) and \( d \) are complex numbers, such that \( ad - bc \neq 0 \).
2. (10pts) (i) State DeMoivre’s theorem.

\[(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta,\]
where \(\theta\) is a real number and \(n\) is positive integer.

(ii) Find formulas for \(\cos 4\theta\) and \(\sin 4\theta\), involving only \(\cos \theta\) and \(\sin \theta\).

We apply DeMoivre’s theorem with \(n = 4\):
\[
\cos 4\theta + i \sin 4\theta = (\cos \theta + i \sin \theta)^4
= \cos^4 \theta + 4i \cos^3 \theta \sin \theta - 6 \cos^2 \theta \sin^2 \theta - 4i \cos \theta \sin^3 \theta + \sin^4 \theta.
\]

Equating real and imaginary parts we get
\[
\cos 4\theta = \cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta
\]
\[
\sin 4\theta = 4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta.
\]
3. (10pts) Write down a Möbius transformation that takes $0$ to $1$, $1$ to $1+i$ and $\infty$ to $2$.

Since $\infty$ goes to $2$ the ratio between $a$ and $c$ is $2$. It follows that neither $a$ nor $c$ is zero. Dividing through by $c$, we may assume that $c = 1$ and $a = 2$, so that we have something of the form

$$z \longrightarrow \frac{2z + b}{z + d}$$

Since $0$ goes to $1$ the ratio between $b$ and $d$ is $1$ so that we have something of the form

$$z \longrightarrow \frac{2z + b}{z + b}$$

The condition that $1$ goes to $1+i$ implies that

$$\frac{2 + b}{1 + b} = 1 + i.$$ 

Thus

$$2 + b = (1 + b)(1 + i) \quad \text{and so} \quad ib = 2 - i - 1 = 1 - i$$

It follows that $b = -1 - i$.

The Möbius transformation

$$z \longrightarrow \frac{2z - 1 - i}{z - 1 - i}$$

takes $0$ to $1$, $1$ to $1+i$ and $\infty$ to $2$. 
4. (15pts) The functions
\[ \sinh : \mathbb{C} \rightarrow \mathbb{C} \quad \text{and} \quad \cosh : \mathbb{C} \rightarrow \mathbb{C} \]
are defined by
\[ \cosh(z) = \cos(iz) \quad \text{and} \quad \sinh(z) = -i \sin(iz). \]
(a) Show that
\[ \sin z = \sin x \cosh y + i \cos x \sinh y \]
(You may use the addition formulae without proof).

The addition formula for sine reads
\[ \sin(z + w) = \cos z \sin w + \sin z \cos w, \]
where \( z \) and \( w \) are complex numbers.
We have
\[
\begin{align*}
\sin z &= \sin(x + iy) \\
&= \cos x \sin(iy) + \sin x \cos iy \\
&= \sin x \cosh y + i \cos x \sinh y.
\end{align*}
\]
(b) Show that
\[ |\sin z|^2 = \sin^2 x + \sinh^2 y \]

Note that
\[ \cosh^2 x + \sinh^2 x = \cos^2(ix) + \sin^2(ix) \]
\[ = 1. \]

It follows that we have
\[ |\sin z|^2 = (\sin x \cosh y)^2 + (\cos x \sinh y)^2 \]
\[ = \sin^2 x \cosh^2 y + \cos 2x \sinh^2 y \]
\[ = \sin^2 x (1 - \sinh^2 y) + \cos^2 x \sinh^2 y \]
\[ = \sin^2 x + (\cos^2 x + \sin^2 x) \sinh^2 y \]
\[ = \sin^2 x + \sinh^2 y. \]

(c) Find all zeroes of the sine function, that is, find all solutions of \( \sin z = 0 \).

Note that
\[ \sin z = 0 \quad \text{if and only if} \quad \sin^2 x + \sinh^2 y = 0. \]

But a sum of squares is zero if and only if each term is zero. If
\[ \sin x = 0 \quad \text{and} \quad \sinh y = 0, \]
then we have \( x \) is a multiple of \( \pi \) and \( y = 0 \).
So the zeroes of \( \sin z \) are just the integer multiples of \( \pi \).
Bonus Challenge Problems

5. (10pts) Prove the triangle inequality

\[ |z + w| \leq |z| + |w|, \]

with equality if and only if either \( z = 0 \) or \( w \) is a positive real scalar multiple of \( z \).

Suppose that \( z + w = re^{i\theta} \). We have

\[
|z + w| = r \\
= e^{-i\theta} (z + w) \\
= \text{Re}(e^{-i\theta} (z + w)) \\
= \text{Re}(e^{-i\theta} z) + \text{Re}(e^{-i\theta} w) \\
\leq |z| + |w|.
\]

Note that we get equality if and only if

\[ \text{Re}(e^{-i\theta} z) = |z| \quad \text{and} \quad \text{Re}(e^{-i\theta} w) = |w|. \]

This happens only if both

\[ e^{-i\theta} z \quad \text{and} \quad e^{-i\theta} w \]

are real. But then \( w \) and \( z \) are real scalar multiples of each other and for equality this multiple has to be non-negative.
6. (10pts) Given three distinct points \( p, q \) and \( r \) of the extended complex plane (so that \( p, q \) and \( r \) are either complex numbers or \( \infty \)) show that there is a unique Möbius transformation

\[
    z \rightarrow \frac{az + b}{cz + d}
\]

taking \( p \) to 0, \( q \) to 1 and \( r \) to \( \infty \).

We break this problem into pieces by writing the Möbius transformation as a composition. The first step is to send \( r \) to \( \infty \) (if it is not already there). The transformation

\[
    z \rightarrow \frac{1}{z - r}
\]

has this property.
Now let us send \( p \) to 0 and at the same time fix \( \infty \). Möbius transformations that fix \( \infty \) look like

\[
    z \rightarrow az + b.
\]

The transformation

\[
    z \rightarrow z - p,
\]

fixes \( \infty \) and sends \( p \) to 0. So now \( p \) and \( r \) are where we want them and we just have to send \( q \) to 1, fixing 0 and \( \infty \). As transformations fixing \( \infty \) look like

\[
    z \rightarrow az + b
\]

transformations that fix 0 and \( \infty \) look like

\[
    z \rightarrow az.
\]

If we want \( q \) to go to 1, we let \( a = 1/q \) to get

\[
    z \rightarrow z/q.
\]

This establishes existence. Observe that if \( M_1 \) and \( M_2 \) are two Möbius transformations sending \( p, q \) and \( r \) to 0, 1 and \( \infty \) then the composition

\[
    M_2 \circ M_1^{-1}
\]

is a Möbius transformation that sends 0, 1 and \( \infty \) to 0, 1 and \( \infty \).
We already know that to fix 0 and \( \infty \) the transformation must be of the form

\[
    z \rightarrow az
\]

and to fix 1 means \( a = 1 \). Thus the composition \( M_1 \circ M_2^{-1} \) is the identity and so it follows that \( M_1 = M_2 \).