MODEL ANSWERS TO THE FIRST HOMEWORK

1. We saw in the previous homework that a circle of radius $\rho$ and centred at the origin is given by the equation

$$|z - a| = \rho.$$ 

Squaring both sides we get

$$\rho^2 = |z - a|^2$$

$$= (z - a)(\bar{z} - a)$$

$$= (z - a)(\bar{z} - \bar{a})$$

$$= z\bar{z} + \bar{a} - a\bar{z} + a\bar{a}$$

$$= |z|^2 + z\bar{a} - a\bar{z} + |a|^2.$$ 

But

$$2 \Re(\bar{a}z) = \bar{a}z + \bar{a}\bar{z}$$

$$= \bar{a}z + a\bar{z}.$$ 

Putting this together gives the result.

2. (a) We have

$$p(i) = i^3 + i^2 + i + 1$$

$$= -i + 1 + i + 1$$

$$= 0.$$ 

(b) There are any number of ways to proceed. $p(z)$ is a real polynomial and so the complex conjugate of $i$, $-i$ is another root. We might then guess that $-1$ is the third root.

Aliter: If the other roots are $\alpha$ and $\beta$ then we know

$$z^3 + z^2 + z + 1 = (z - i)(z - \alpha)(z - \beta).$$ 

Multiplying out the RHS and equating coefficients gives us

$$-i\alpha\beta = 1$$

and

$$-i - \alpha - \beta = 1,$$ 

so that

$$\alpha\beta = i$$

and

$$\alpha + \beta = -1 - i.$$ 

Thus $\alpha$ and $\beta$ are the roots of the quadratic polynomial

$$z^2 + (1 + i)z - i.$$ 

Now complete the square or use the quadratic formula.
Aliter: We can do long division and divide the linear factor \( z + i \) into the polynomial \( p(z) \). We know we won’t get a remainder and the quotient is in fact
\[
z^2 + (1 + i)z - i.
\]

Aliter: If we multiply \( p(z) \) by the polynomial \( z - 1 \) we get the polynomial
\[
z^4 - 1.
\]
The roots are the fourth roots of unity. \( i \) is a fourth root of unity and \( 1 \) is a root of \( z - 1 \). What is left are \(-1 \) and \(-i\) and these are the other roots.

3. We have
\[
i = i \quad i^2 = -1 \quad i^3 = -i \quad \text{and} \quad i^4 = 1.
\]
Thus the powers of \( i \) are periodic with period 4. \( i \) is an \( n \)th root of unity if and only if \( n \) is divisible by 4.

4. Let \( n \geq 1 \) be an integer.
(a) There are three ways to proceed. The easiest is to treat \( z \) as a variable. It is clear that
\[
(1 + z + z^2 + z^3 + \cdots + z^n)(1 - z) = 1 - z^{n+1}
\]
and dividing through by \( 1 - z \) gives the result.

Aliter: We could use induction on \( n \). The result is clear if \( n = 0 \), since the LHS is 1 and the RHS is
\[
\frac{1 - z}{1 - z}.
\]
Assume the result for \( n \) and let’s see what happens for \( n + 1 \)
\[
1 + z + z^2 + z^3 + \cdots + z^n + z^{n+1} = (1 + z + z^2 + z^3 + \cdots + z^n) + z^{n+1}
\]
\[
= \frac{1 - z^{n+1}}{1 - z} + z^{n+1}
\]
\[
= \frac{1 - z^{n+1} + (1 - z)z^{n+1}}{1 - z}
\]
\[
= \frac{1 - z^{n+2}}{1 - z}.
\]
This completes the induction and the proof.

Aliter: We could recognize that we have a geometric series with common ratio \( z \) and use the trick of Gauss. Call the sum on the LHS.
Multiplying by $z$ gives us:

\[ S = 1 + z + z^2 + z^3 + \cdots + z^n \]

\[ zS = z + z^2 + z^3 + \cdots + z^n + z^{n+1} \]

As the expressions on the RHS have so many common terms it makes sense to subtract:

\[ (1 - z)S = 1 - z^{n+1}. \]

Dividing gives the result.

(b) We apply (a) with $z = e^{i\theta}$. We get

\[ 1 + e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta} = \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}. \]

Now we equate the real parts. The real part of the LHS is

\[ 1 + \cos \theta + \cos 2\theta + \cdots + \cos n\theta. \]

For the RHS, we first attack the denominator:

\[ 1 - e^{i\theta} = e^{i\theta/2}(e^{-i\theta/2} - e^{i\theta/2}) = -2ie^{i\theta/2}\sin \theta/2. \]

Note that the reciprocal of $-ie^{i\theta/2}$ is $ie^{-i\theta/2}$.

Thus the RHS is

\[ \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} = \frac{ie^{-i\theta/2} - ie^{i(n+1/2)\theta}}{2\sin \theta/2}. \]

Taking the real part we get

\[ \frac{\sin \theta/2 + \sin(n + 1/2)\theta}{2\sin \theta/2} = \frac{1}{2} + \frac{\sin(n + \frac{1}{2})\theta}{2\sin \frac{\theta}{2}}. \]

5. Multiplying top and bottom by $\cos \theta$ we get

\[ \left(\frac{1 + i \tan \theta}{1 - i \tan \theta}\right)^n = \left(\frac{\cos \theta + i \sin \theta}{\cos \theta - i \sin \theta}\right)^n \]

\[ = \left(\frac{e^{i\theta}}{e^{-i\theta}}\right)^n \]

\[ = (e^{2i\theta})^n \]

\[ = e^{i2n\theta} \]

\[ = \frac{1 + i \tan n\theta}{1 - i \tan n\theta}. \]
To get from the penultimate line to the last line we use the identity established to get from the first line to the third line.

6. We want to solve
\[ z^6 = -64. \]

If we put
\[ z = re^{i\theta} \]
then we get the equation:
\[ r^6e^{6i\theta} = -64. \]

Taking the modulus of both sides we get
\[ r = 2. \]

Cancelling we are reduced to solving:
\[ e^{6i\theta} = -1 = e^{i\pi}. \]

One solution is
\[ 6\theta = \pi \quad \text{so that} \quad \theta = \frac{\pi}{6}. \]

But we might go once around the circle so that another solution is
\[ 6\theta = \pi + 2\pi \quad \text{so that} \quad \theta = \frac{\pi}{2}. \]

Continuing in this way gives us all six solutions;
\[ 6\theta = 5\pi \quad \text{so that} \quad \theta = \frac{5\pi}{6}; \]
\[ 6\theta = 7\pi \quad \text{so that} \quad \theta = \frac{7\pi}{6}; \]
\[ 6\theta = 9\pi \quad \text{so that} \quad \theta = \frac{3\pi}{2}; \]
\[ 6\theta = 11\pi \quad \text{so that} \quad \theta = \frac{11\pi}{6}. \]

The sixth roots of $-1$ are therefore
\[ e^{i\pi/6}; \quad e^{i\pi/2}; \quad e^{5i\pi/6}; \quad e^{7i\pi/6}; \quad e^{3i\pi/2}; \quad \text{and} \quad e^{11i\pi/6}. \]

The sixth roots of $-64$ are
\[ 2e^{i\pi/6}; \quad 2e^{i\pi/2}; \quad 2e^{5i\pi/6}; \quad 2e^{7i\pi/6}; \quad 2e^{3i\pi/2}; \quad \text{and} \quad 2e^{11i\pi/6}. \]

There is an interesting connection between this problem and the problem of finding the twelfth roots of unity. If
\[ \zeta = e^{i\pi/6} \]
then the powers of $\zeta$ are 12th roots of unity. The even powers are sixth roots of unity but the odd powers are sixth roots of $-1$. Thus we just want the odd powers of $\zeta$: $\zeta, \zeta^3, \zeta^5, \zeta^7, \zeta^9,$ and $\zeta^{11}$.

7. We first put $1 - \sqrt{3}i$ into polar form

$$1 - \sqrt{3}i = 2e^{i\pi/3}.$$ 

It follows that

$$(1 - \sqrt{3}i)^{10} = (2e^{i2\pi/3})^{10}$$
$$= 2^{10}e^{-i20\pi/3}$$
$$= 2^{10}e^{-i4\pi/3}$$
$$= 2^{10}e^{i2\pi/3}$$
$$= 2^9(-1 + \sqrt{3}i).$$

**Challenge Problems:** (Just for fun)

8. Suppose that $z + w = re^{i\theta}$. We have

$$|z + w| = r$$
$$= e^{-i\theta}(z + w)$$
$$= \text{Re}(e^{-i\theta}(z + w))$$
$$= \text{Re}(e^{-i\theta}z) + \text{Re}(e^{-i\theta}w)$$
$$\leq |z| + |w|.$$ 

Note that we get equality if and only if

$$\text{Re}(e^{-i\theta}z) = |z|\quad\text{and}\quad\text{Re}(e^{-i\theta}w) = |w|.$$ 

This happens only if both $e^{-i\theta}z$ and $e^{-i\theta}w$ are real. But then $w$ and $z$ are real scalar multiples of each other and for equality this multiple has to be non-negative.