MODEL ANSWERS TO THE FIFTH HOMEWORK

1. (a) This is a polynomial and so holomorphic. We apply the chain rule:

$$\begin{aligned} f'(z) &= \frac{\mathrm{d}}{\mathrm{d}z}(2z^2+1)^5 \\ &= 10z(2z^2+1)^4. \end{aligned}$$

(b) This is a rational function, a ratio of two polynomials. It is therefore holomorphic. We apply the quotient rule:

$$\begin{aligned} f'(z) &= \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z-1}{2z+1}\right) \\ &= \frac{1(2z-1)-(z-1)2}{(2z+1)^2} \\ &= \frac{1}{(2z+1)^2}. \end{aligned}$$

(c) This is also a rational function, and so holomorphic. We apply the quotient and chain rule:

$$\begin{aligned} f'(z) &= \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{(1+z^2)^4}{z^2}\right) \\ &= \frac{8z(1+z^2)^3z^2 - 2z(1+z^2)^4}{z^4} \\ &= 2(1+z^2)^3\frac{4z^2-(1+z^2)}{z^3} \\ &= \frac{2(3z^2-1)(1+z^2)^3}{z^3}. \end{aligned}$$

(d) This is the quotient of two holomorphic functions and so holomorphic. We apply the quotient rule

$$\begin{aligned} f'(z) &= \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\sin z}{\cos z}\right) \\ &= \frac{\cos z\cos z + \sin z\sin z}{\cos^2 z} \\ &= \frac{1}{\cos^2 z} \\ &= \sec^2 z. \end{aligned}$$

(e) $\cosh(z) = \cos(iz)$ and so $f(z)$ is holomorphic, as the composition of holomorphic functions is holomorphic. We apply the chain rule

$$\begin{aligned} f'(z) &= \frac{\mathrm{d}}{\mathrm{d}z}\cos(iz) \\ &= -i\sin(iz) \\ &= \sinh(z). \end{aligned}$$

(f) $\sinh(z) = -i\sin(iz)$ and so $f(z)$ is holomorphic, as the composition of holomorphic functions is holomorphic. We apply the chain rule

$$\begin{aligned} f'(z) &= \frac{\mathrm{d}}{\mathrm{d}z}(-i\sin(iz)) \\ &= \cos(iz) \\ &= \cosh(z). \end{aligned}$$

2. (a) We have to show the limit

$$\lim_{z\to a}\frac{f(z)-f(a)}{z-a}$$

does not exist. We have

$$\begin{aligned} \frac{f(z)-f(a)}{z-a} &= \frac{\bar{z}-\bar{a}}{z-a} \\ &= \frac{\overline{z-a}}{z-a}. \end{aligned}$$

If the difference $z-a$ is real then

$$\overline{z-a} = z-a.$$

In this case

$$\begin{aligned} \lim_{z\to a}\frac{f(z)-f(a)}{z-a} &= \lim_{z\to a}\frac{z-a}{z-a} \\ &= \lim_{z\to a} 1 \\ &= 1. \end{aligned}$$

On the other hand, if the difference $z-a$ is imaginary then

$$\overline{z-a} = -(z-a).$$

In this case

$$\begin{aligned} \lim_{z\to a}\frac{f(z)-f(a)}{z-a} &= \lim_{z\to a} -\frac{z-a}{z-a} \\ &= \lim_{z\to a} -1 \\ &= -1. \end{aligned}$$

Thus the limit does not exist as you don't get the same answer if you approach along a different line.
(b) We have $u(x,y) = x$ and $v(x,y) = -y$. We have

$$\begin{aligned} u_x &= 1 \\ &\neq -1 \\ &= v_y. \end{aligned}$$

Thus $z \longrightarrow \bar{z}$ does not satisfy the Cauchy-Riemann equations and so it is not holomorphic.
3. (a) We have

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta.$$

It follows by the chain rule for partial derivatives that

$$\begin{aligned} \frac{\partial u}{\partial r} &= \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y} \\ &= \cos\theta\frac{\partial v}{\partial y} - \sin\theta\frac{\partial v}{\partial x} \\ &= \frac{1}{r}\left(-r\sin\theta\frac{\partial v}{\partial x} + r\cos\theta\frac{\partial v}{\partial y}\right) \\ &= \frac{1}{r}\frac{\partial v}{\partial \theta} \end{aligned}$$

Simliarly

$$\begin{aligned} \frac{\partial u}{\partial \theta} &= -r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y} \\ &= -r\left(\sin\theta\frac{\partial u}{\partial x} - \cos\theta\frac{\partial u}{\partial y}\right) \\ &= -r\left(\cos\theta\frac{\partial v}{\partial x} + \sin\theta\frac{\partial v}{\partial y}\right) \\ &= -r\frac{\partial v}{\partial r}. \end{aligned}$$

(b) We have

$$\begin{aligned} \frac{\partial u}{\partial r} &= mr^{m-1}\cos(m\theta) & \qquad \frac{\partial u}{\partial \theta} &= -mr^m\sin(m\theta) \\ \frac{\partial v}{\partial r} &= mr^{m-1}\sin(m\theta) & \qquad \frac{\partial v}{\partial \theta} &= mr^m\cos(m\theta). \end{aligned}$$

It follows that

$$\begin{aligned}\frac{\partial u}{\partial r} &= mr^{m-1}\cos(m\theta) \\ &= \frac{1}{r}mr^m\cos(m\theta) \\ &= \frac{1}{r}\frac{\partial v}{\partial \theta}.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial u}{\partial \theta} &= -mr^m\sin(m\theta) \\ &= -rmr^{m-1}\sin(m\theta) \\ &= -r\frac{\partial v}{\partial r}.\end{aligned}$$

4. Let $w = u+iv$ be the coordinates on the image $f(U)$. Then the area of the image is the volume under the graph of the constant function 1:

$$\iint_{f(U)} \mathrm{d}u\mathrm{d}v.$$

We have

$$u + iv = f(x + iy).$$

If we change variables then we have to multiply through by the Jacobian

$$\begin{aligned}\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} &= u_x v_y - u_y v_x \\ &= u_x u_x + v_x v_x \\ &= u_x^2 + v_x^2 \\ &= |f'(z)|^2.\end{aligned}$$

Thus the area of $f(U)$ is

$$\iint_{f(U)} \mathrm{d}u\mathrm{d}v = \iint_U |f'(z)|^2 \,\mathrm{d}x\mathrm{d}y,$$

by change of variables.

Challenge Problems: (Just for fun)

5. (a) The solutions of

$$\sin(\pi z) = 0$$

are the integers, $0, \pm 1, \pm 2, \ldots$.

(b) An infinite product expansion for $\sin(\pi z)$ is given by

$$\sin \pi z = \alpha z\,(1 - z)\,(1 + z)\left(1 - \frac{z}{2}\right)\left(1 + \frac{z}{2}\right)\ldots,$$

where α is a complex number. If we differentiate the LHS we get $\pi \cos \pi z$. If we plug in $z = 0$ then we get π. The RHS is a product of αz and what is left, call it v. If we differentiate the RHS, apply Leibniz and plug in $z = 0$, the only term that survives is $\alpha v(0)$. $v(0) = 1$ and so the RHS is α. Thus $\alpha = \pi$,

$$\sin \pi z = \pi z\,(1 - z)\,(1 + z)\left(1 - \frac{z}{2}\right)\left(1 + \frac{z}{2}\right)\ldots,$$

(c) Grouping positive and negative roots together, we get

$$\sin(\pi z) = \pi z\left(1 - \frac{z^2}{1}\right)\left(1 - \frac{z^2}{4}\right)\left(1 - \frac{z^2}{9}\right),\ldots.$$

(d) We have

$$\begin{aligned}\sin(\pi z) &= \pi z\left(1 - \frac{z^2}{1}\right)\left(1 - \frac{z^2}{4}\right)\left(1 - \frac{z^2}{9}\right),\ldots\\ &= \pi z - \pi a_3 z^3 + \pi a_5 z^5 + \ldots.\end{aligned}$$

(e) We have

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots.$$

Replacing z by πz gives

$$\sin z = \pi z - \frac{\pi^3 z^3}{3!} + \frac{\pi^5 z^5}{5!} - \ldots.$$

The coefficient of z^3 is

$$-\frac{\pi^3}{6}.$$

It follows that

$$\pi a_3 = \frac{\pi^3}{6}.$$

It remains to compute a_3. To get a coefficient of z^3, we have to pull z^2 from one bracket and 1 from every other bracket. In total we get

$$a_3 = 1 + \frac{1}{4} + \frac{1}{9} + \cdots = \zeta(2).$$

Thus

$$\pi\zeta(2) = \frac{\pi^3}{6}.$$

It follows that

$$\zeta(2) = \frac{\pi^2}{6}.$$