MODEL ANSWERS TO THE SIXTH HOMEWORK

1. (a) Suppose that $f(z) = u(x, y) + iv(x, y)$. If the real part of f is constant then u is constant and so $u_x = u_y = 0$ on U. As f is holomorphic it satisfies the Cauchy-Riemann equations. But then

$$\begin{aligned} v_y &= u_x \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} v_x &= -u_y \\ &= 0. \end{aligned}$$

It follows that v is constant. But then f is constant.
(b) Let $g = if$. g is holomorphic as f is holomorphic. As the imaginary part of f is constant it follows that the real part of g is constant. By part (a) g is constant. But then f is constant.
2. We have to compute the following limit (if it exists at all)

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a}.$$

As a first step let us manipulate the numerator.

$$\begin{aligned} f(z) - f(a) &= \int_0^1 \frac{h(t)}{t - z} \, \mathrm{d}t - \int_0^1 \frac{h(t)}{t - a} \, \mathrm{d}t \\ &= \int_0^1 \frac{h(t)}{t - z} - \frac{h(t)}{t - a} \, \mathrm{d}t \\ &= \int_0^1 \frac{h(t)(t - a) - h(t)(t - z)}{(t - z)(t - a)} \, \mathrm{d}t \\ &= \int_0^1 \frac{h(t)(z - a)}{(t - z)(t - a)} \, \mathrm{d}t \\ &= (z - a) \int_0^1 \frac{h(t)}{(t - z)(t - a)} \, \mathrm{d}t. \end{aligned}$$

If we divide through by $z - a$ we get

$$\int_0^1 \frac{h(t)}{(t - z)(t - a)} \, \mathrm{d}t.$$

If we take the limit as z approaches a we get

$$\int_0^1 \frac{h(t)}{(t-a)^2}\,\mathrm{d}t$$

(this is a uniform limit as a is at least a fixed distance from the interval $[0,1]$). It follows that the limit exists, so that f is a holomorphic function and the derivative at a is

$$\int_0^1 \frac{h(t)}{(t-a)^2}\,\mathrm{d}t.$$

3. (a) We have

$$\begin{aligned}\frac{1}{z} &= \frac{1}{x+iy}\\ &= \frac{x-iy}{x^2+y^2}.\end{aligned}$$

It follows that

$$u = \frac{x}{x^2+y^2} \qquad \text{and} \qquad v = \frac{-y}{x^2+y^2}.$$

$1/z$ is holomorphic everywhere, except at the origin. Its derivative is nowhere zero and so it is conformal on $U = \mathbb{C} \setminus \{0\}$.
(b) We have

$$\begin{aligned}\frac{1}{z^2} &= \frac{1}{(x+iy)^2}\\ &= \frac{(x-iy)^2}{(x^2+y^2)^2}\\ &= \frac{x^2-y^2-2ixy}{(x^2+y^2)^2}.\end{aligned}$$

It follows that

$$u = \frac{x^2-y^2}{(x^2+y^2)^2} \qquad \text{and} \qquad v = \frac{-2xy}{(x^2+y^2)^2}.$$

$1/z^2$ is holomorphic everywhere, except at the origin. Its derivative is nowhere zero and so it is conformal on $U = \mathbb{C} \setminus \{0\}$.
(c) We have

$$\begin{aligned}z^6 &= (x+iy)^6\\ &= x^6 + 6ix^5y - 15x^4y^2 - 20ix^3y^3 + 15x^2y^4 + 6ixy^5 - y^6\\ &= x^6 - 15x^4y^2 + 15x^2y^4 - y^6 + i(6x^5y - 20x^3y^3 - 6xy^5).\end{aligned}$$

It follows that

$$u = x^6 - 15x^4y^2 + 15x^2y^4 - y^6 \qquad \text{and} \qquad v = 6x^5y - 20x^3y^3 - 6xy^5.$$

z^6 is holomorphic everywhere. Its derivative is zero at zero and not zero anywhere else, and so it is conformal on $U = \mathbb{C} \setminus \{0\}$.

4. Pick two different values of r, r_1 and $r_2 \in (a, b)$. We just have to show that the two integrals are equal. We may assume that $r_1 < r_2$. Let U be the region bounded between the two circles, another annulus. Then the boundary of U is the two circles γ_{r_1} and γ_{r_2} and so $U \cup \partial U \subset V$. It follows that we may apply Green's theorem. Note that the boundary of U consists of two circles

$$\gamma_1 = -\gamma_{r_1} \qquad \text{and} \qquad \gamma_2 = \gamma_{r_2}.$$

This minus sign in front of γ_{r_1} is meant to indicate that we traverse the circle γ_{r_1} clockwise, the opposite direction to usual. This has the effect of flipping the sign of the integral.

Green's theorem says

$$\begin{aligned} \int_{\gamma_{r_2} - \gamma_{r_1}} P\,\mathrm{d}x + Q\,\mathrm{d}y &= \int_{\gamma_2 + \gamma_1} P\,\mathrm{d}x + Q\,\mathrm{d}y \\ &= \int_{\partial U} P\,\mathrm{d}x + Q\,\mathrm{d}y \\ &= \iint_U \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x\mathrm{d}y \\ &= \iint_U 0\mathrm{d}x\mathrm{d}y \\ &= 0. \end{aligned}$$

It follows that

$$\int_{\gamma_{r_2}} P\,\mathrm{d}x + Q\,\mathrm{d}y + \int_{-\gamma_{r_1}} P\,\mathrm{d}x + Q\,\mathrm{d}y = 0,$$

so that

$$\int_{\gamma_{r_2}} P\,\mathrm{d}x + Q\,\mathrm{d}y = \int_{\gamma_{r_1}} P\,\mathrm{d}x + Q\,\mathrm{d}y.$$

5. In all three cases we use the parametrisation

$$t \longrightarrow z = 2e^{it}.$$

In this case

$$\begin{aligned} \frac{z+2}{z} &= 1 + \frac{2}{z} \\ &= 1 + e^{-it}, \end{aligned}$$

on the boundary. On the other hand

$$\mathrm{d}z = 2ie^{it}\,\mathrm{d}t.$$

It follows that

$$\frac{z+2}{z}\,\mathrm{d}z = 2i(1+e^{it})\,\mathrm{d}t.$$

(a) We have $\gamma_1(t) = 2e^{it}$, where $t \in [0, \pi]$. Thus

$$\begin{aligned}\int_{\gamma_1} \frac{z+2}{z}\,\mathrm{d}z &= \int_0^{\pi} 2i(1+e^{it})\,\mathrm{d}t \\ &= \left[2it + 2e^{it}\right]_0^{\pi} \\ &= 2\pi i - 4.\end{aligned}$$

(b) We have $\gamma_2(t) = 2e^{it}$, where $t \in [\pi, 2\pi]$. Thus

$$\begin{aligned}\int_{\gamma_2} \frac{z+2}{z}\,\mathrm{d}z &= \int_{\pi}^{2\pi} 2i(1+e^{it})\,\mathrm{d}t \\ &= \left[2it + 2e^{it}\right]_{\pi}^{2\pi} \\ &= 2\pi i + 4.\end{aligned}$$

(c) We have $\gamma_3(t) = 2e^{it}$, where $t \in [0, 2\pi]$. As

$$\gamma_3 = \gamma_1 + \gamma_2,$$

it follows that

$$\begin{aligned}\int_{\gamma_3} \frac{z+2}{z}\,\mathrm{d}z &= \int_{\gamma_1+\gamma_2} \frac{z+2}{z}\,\mathrm{d}z \\ &= \int_{\gamma_1} \frac{z+2}{z}\,\mathrm{d}z + \int_{\gamma_2} \frac{z+2}{z}\,\mathrm{d}z \\ &= 2\pi i - 4 + 2\pi i + 4 \\ &= 4\pi i.\end{aligned}$$

6. We want to apply Green's theorem to compute the line integral. If $\gamma = \partial U$ then the integrand of the line integral is

$$\begin{aligned}\bar{z}\,\mathrm{d}z &= (x - iy)(\mathrm{d}x + i\mathrm{d}y) \\ &= x\mathrm{d}x + y\mathrm{d}y + i(-y\mathrm{d}x + x\mathrm{d}y) \\ &= (x - iy)\mathrm{d}x + (y + ix)\mathrm{d}y \\ &= P\mathrm{d}x + Q\mathrm{d}y.\end{aligned}$$

Note that

$$\frac{\partial P}{\partial y} = -i \qquad \text{and} \qquad \frac{\partial Q}{\partial x} = i.$$

Green's theorem says

$$\begin{aligned}\int_\gamma \bar{z}\,\mathrm{d}z &= \int_{\partial U} P\,\mathrm{d}x + Q\,\mathrm{d}y \\ &= \iint_U \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\mathrm{d}y \\ &= \iint_U 2i \mathrm{d}x\mathrm{d}y \\ &= 2i \iint_U \mathrm{d}x\mathrm{d}y.\end{aligned}$$

On the other hand

$$\iint_U \mathrm{d}x\mathrm{d}y$$

is the volume under the graph of the constant function 1, which is the area of U.

Challenge Problems: (Just for fun)

1. (contd) As U is connected, we may prove this locally on U. Possibly multiplying f by a constant we may assume f is nowhere real. In this case we can compose with the principal value of the logarithm, to get a holomorphic function

$$g(z) = \mathrm{Log}(f(z)).$$

If $f(z) = re^{i\theta}$ then

$$g(z) = \ln r + i\theta,$$

where θ is the principal value of the argument.
(c) If the modulus of f is constant then r is constant. It follows that the real part of g is constant. By part (a) it follows that g is constant. But then f is constant.
(d) If the argument of f is constant then θ is constant. It follows that the imaginary part of g is constant. By part (b) it follows that g is constant. But then f is constant.
7. (a) We are free to apply a translation and so may assume that $a = 0$. After that we may rotate until b is a real number. Finally we can rescale so that $b = 1$.
So we are looking at the set of points such that

$$l\sqrt{(x^2 + y^2)} = \sqrt{((x-1)^2 + y^2)}.$$

Squaring both sides, expanding and rearranging gives

$$(1 - l^2)x^2 - 2x + 1 + (1 - l^2)y^2 = 0$$

Dividing through by $1 - l^2$, we get

$$x^2 + y^2 - \frac{2}{x} 1 - l^2 + \frac{1}{1 - l^2} = 0.$$

Completing the square we get the equation of a circle.
(b) Pick a Möbius transformation that fixes 0 and 1 and sends a point of the circle C_1 to ∞. The circle C_1 becomes the perpendicular bisector of 0 and 1, that is, the line $x = 1/2$. On the other hand, C_2 is still a circle through 0 and 1.
It easy to see that C_2 is orthogonal to the line $x = 1/2$.