1. The function
\[
\frac{1}{z^2 - z}
\]
is not holomorphic at \( z = 0 \) and at \( z = 1 \). There are two relevant circles, the circle of radius 1 centred at \(-1\) and the circle of radius 2 centred at \(-1\). The point \( 1/2 \) is between these two circles, so we want to compute the Laurent series for the annulus

\[
U = \{ z \in \mathbb{C} \mid 1 < |z + 1| < 2 \}.
\]

We have
\[
\frac{1}{z^2 - z} = \frac{1}{z(z - 1)} = \frac{1}{z - 1} - \frac{1}{z}.
\]
The first function is holomorphic on the disk of radius 2 centred at \(-1\).

The second function is holomorphic on the region \(|z + 1| > 1\) and is zero at infinity. We have
\[
\frac{1}{z - 1} = \frac{1}{-2 + z + 1} = -\frac{1}{2} \frac{1}{1 - (z + 1)/2} = -\frac{1}{2} \frac{z + 1}{4} - \frac{(z + 1)^2}{8} + \ldots.
\]
This is a power series centred at \( z = -1 \) with radius of convergence 2. For the second function we have
\[
-\frac{1}{z} = -\frac{1}{-1 + (z + 1)} = \frac{1}{z + 1 - 1/(z + 1)} = \frac{1}{z + 1} + \frac{1}{(z + 1)^2} + \frac{1}{(z + 1)^3} + \ldots.
\]
Thus
\[ \frac{1}{z^2 - z} = \frac{1}{z - 1} - \frac{1}{z} \]
\[ = \cdots + \frac{1}{(z + 1)^3} + \frac{1}{(z + 1)^2} + \frac{1}{z + 1} - \frac{1}{2} - \frac{z + 1}{4} - \frac{(z + 1)^2}{8} + \cdots \]
is the Laurent expansion.

(b) The function
\[ \frac{z - 1}{z + 1} \]
is not holomorphic at \( z = -1 \). There are two relevant circles, the circle of radius 0 centred at \(-1\) and the circle of radius \( \infty \) centred at \(-1\). The point \( 1/2 \) is between these two circles, so we want to compute the Laurent series for the annulus
\[ U = \{ z \in \mathbb{C} \mid 0 < |z + 1| < \infty \}. \]

We have
\[ z - 1 \]
\[ z + 1 \]
\[ = \frac{z + 1 - 2}{z + 1} \]
\[ = -\frac{2}{z + 1} + 1. \]

This is a Laurent expansion and so it is the Laurent expansion.

2. (a) We have
\[ \frac{1}{z + z^2} = \frac{1}{z(z + 1)}. \]
This has a simple pole at \( z = 0 \).
\[ \text{Res}_0 \frac{1}{z + z^2} = \lim_{z \to 0} \frac{1}{z + 1} = 1. \]

(b) We have
\[ z \cos \left( \frac{1}{z} \right) = z \left( 1 - \frac{1}{2z^2} + \frac{1}{4!z^4} + \cdots \right) \]
\[ = z - \frac{1}{2z} + \frac{1}{4!z^3} + \cdots. \]
It follows that the residue at \( z = 0 \) is \(-1/2\).

(c) As \( \sinh z \) has a simple zero at 0 it follows that
\[ \frac{\sinh z}{z^4(1 - z^2)} \]
has a pole of order 3 at 0. We could try multiplying by \( z^3 \) and differentiating twice to get the residue; this doesn’t seem to work very well.

If we expand the power series for \( \sinh z \) and for the reciprocal of \( 1 - z^2 \) we get:

\[
\frac{\sinh z}{z^4(1 - z^2)} = \frac{1}{z^4} \left( z + \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots \right) \left( 1 + z^2 + z^4 + \ldots \right)
\]

We want the coefficient of \( 1/z \) after we multiply out. So we want the coefficient of \( z^3 \) from the second two expressions. This is

\[1 + \frac{1}{3!} = \frac{7}{6}.
\]

Therefore the residue is \( 7/6 \).

3. As \( (z - a)^nf(z) \) is bounded near \( a \) and it has an isolated singularity at \( a \), it follows that \( (z - a)^nf(z) \) has a removable singularity at \( z = a \).

In particular there is a holomorphic function \( g(z) \) such that

\[(z - a)^nf(z) = g(z)\]

Suppose that \( g(z) \) has a zero of order \( m \) at \( a \). Then there is a function \( h(z) \) holomorphic at \( a \) such that

\[g(z) = (z - a)^m h(z) \quad \text{and} \quad h(a) \neq 0.
\]

It follows that

\[(z - a)^nf(z) = (z - a)^m h(z).
\]

If \( m \geq n \) then

\[f(z) = (z - a)^{m-n} h(z)
\]

has a removable singularity at \( a \). Otherwise

\[f(z) = \frac{h(z)}{(z - a)^{n-m}}.
\]

In this case \( f(z) \) has a pole order at most \( n \).

4. (a) We use the residue theorem. The function

\[\frac{z}{\cos z}\]

has isolated singularities at

\[z = \pm \pi/2\]

which are both inside the circle of radius 2, as \( \pi/2 < 2 \). \( \cos z \) has simple singularities at these points and so the function

\[\frac{z}{\cos^3 z}\]
has simple poles at $\pm \pi/2$. To compute the residue at $\pi/2$ we multiply by $(z - \pi/2)$ and take a limit:

$$\text{Res}_{\pi/2} \frac{z}{\cos z} = \lim_{z \to \pi/2} \frac{z(z - \pi/2)}{\cos z}$$

$$= \lim_{z \to \pi/2} \frac{2z - \pi/2}{- \sin z}$$

$$= -\frac{\pi}{2}.$$ 

To compute the residue at $-\pi/2$ we multiply by $(z + \pi/2)$ and take a limit:

$$\text{Res}_{-\pi/2} \frac{z}{\cos z} = \lim_{z \to -\pi/2} \frac{z(z + \pi/2)}{\cos z}$$

$$= \lim_{z \to -\pi/2} \frac{2z + \pi/2}{- \sin z}$$

$$= -\frac{\pi}{2}.$$ 

Now we apply the residue theorem:

$$\oint_{|z|=2} \frac{z}{\cos z} \, dz = 2\pi i \left( \text{Res}_{\pi/2} \frac{z}{\cos z} + \text{Res}_{-\pi/2} \frac{z}{\cos z} \right)$$

$$= 2\pi i \left( -\frac{\pi}{2} - \frac{\pi}{2} \right)$$

$$= -2\pi^2 i.$$ 

(b) The function

$$\frac{e^{-z}}{z^2}$$

has an isolated singularity at zero, which is inside the circle. As the function has a double pole at 0, to compute the residue we multiply by $z^2$ and differentiate once:

$$\text{Res}_0 \frac{e^{-z}}{z^2} = \lim_{z \to 0} -e^{-z}$$

$$= -1.$$ 

Now we apply the residue theorem:

$$\oint_{|z|=3} \frac{e^{-z}}{z^2} \, dz = 2\pi i \text{Res}_0 \frac{e^{-z}}{z^2}$$

$$= -2\pi i.$$ 

(c) The function

$$z^2 e^{1/z}$$
has a singularity at $z = 0$. As we have an essential singularity we simply have to compute the Laurent series:

$$z^2 e^{1/z} = z^2 \left( 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \ldots \right)$$

$$= z^2 + z + \frac{1}{2} + \frac{z}{6} + \ldots$$

Thus the residue is

$$\text{Res}_0 z^2 e^{1/z} = \frac{1}{6}.$$ 

Now we apply the residue theorem:

$$\oint_{|z|=1} z^2 e^{1/z} \, dz = 2\pi i \text{Res}_0 z^2 e^{1/z}$$

$$= \frac{\pi i}{3}.$$