PRACTICE PROBLEMS FOR THE SECOND MIDTERM

1. (a) Give the definition of:
   (i) a power series;
   (ii) the centre of a power series;
   (iii) the radius of convergence of a power series;
   (iv) a (complex) analytic function;
   (v) the Riemann zeta function;
   (vi) (complex) differentiable at a point;
   (vii) a holomorphic function;
   (viii) an entire function;
   (ix) the tangent vector to a curve;
   (x) a conformal map;
   (xi) a line integral;
(b) State
   (i) the Cauchy-Riemann equations;
   (ii) Green’s theorem;
   (iii) Cauchy’s theorem;
   (iv) Cauchy’s integral formula.

2. Show that
   \[ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \]
diverges, whilst
   \[ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \]
converges.

2. Find the first five terms of the power series expansion of
   \[ \frac{e^z}{1 - z} \]
centred at 0. What is the radius of convergence?
3. Where is the following function holomorphic? Find its derivative
   \[ \frac{e^{2z^2}}{z^2 - 5z + 6}. \]
4. Show that the first quadrant
   \[ \{ z \in \mathbb{C} \mid 0 < \text{Arg}(z) < \frac{\pi}{2} \} \]
and the unit circle are conformally equivalent.
5. Write down the polar form of the Cauchy-Riemann equations and check that the functions

\[ u(r, \theta) = r^m \cos(m\theta) \quad \text{and} \quad v(r, \theta) = r^m \sin(m\theta) \]

satisfy these equations.

6. Suppose that \( P \) and \( Q \) are two functions on the annulus

\[ V = \{ \, z \in \mathbb{C} \mid a < |z| < b \, \} \]

which have continuous partial derivatives. If

\[ \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \]

then show that the integral

\[ \int_{\gamma_r} P \, dx + Q \, dy \]

is independent of \( r \), where \( \gamma_r \) is the circle of radius \( r \in (a, b) \) centred at the origin and we traverse \( \gamma_r \) counterclockwise.