Non-trivial d-wise Intersecting Families

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UC San Diego

November 9, 2019

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4. Open Problems and Conjectures

Preliminaries

Let $[n] := \{1, 2, \ldots, n\}$ and $\binom{[n]}{k} := \{A \subset [n] : |A| = k\}$.

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Definition

A family $\mathcal{F} \subset \binom{[n]}{k}$ is said to be d-**wise intersecting** if for all $A_1, \ldots, A_d \in \mathcal{F}$, we have that

$$\bigcap_{i=1}^{d} A_i \neq \emptyset.$$

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In the case where $d = 2$, we say that $\mathcal{F}$ is intersecting.

The Erdős-Ko-Rado Theorem

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Theorem (Erdős-Ko-Rado, 1961)

Let $n \geq 2k$ and $\mathcal{F} \subset \binom{[n]}{k}$ be an intersecting family. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

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Theorem (Erdős-Ko-Rado, 1961)

Let $n \geq 2k$ and $\mathcal{F} \subset \binom{[n]}{k}$ be an intersecting family. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

Moreover, if $n > 2k$ and $|\mathcal{F}| = \binom{n-1}{k-1}$, then $\mathcal{F} \cong \mathcal{A}$.

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Question (Erdős-Ko-Rado, 1961)

If $n \geq 2k$, what is the largest non-trivial intersecting family $\mathcal{F} \subset \binom{[n]}{k}$?

The Hilton-Milner theorem (d=2)

Theorem (Hilton-Milner, 1967)

Let $n > 2k$ and $k \geq 3$. If $\mathcal{F} \subset \binom{[n]}{k}$ is a non-trivial intersecting family, then $|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$.

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We can achieve the upper bound in the above Theorem with

$$\mathcal{HM}(k,2) = \{[2,k+1]\} \cup \{A \in \binom{n}{k} : 1 \in A, A \cap [2,k+1] \neq \emptyset\}.$$

The case when $d > k$

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$$A \cap \bigcap_{a \in A} X_a = \emptyset.$$

□

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Question (Hilton-Milner)

For $2 < d < k$, what is the the largest non-trivial d-wise intersecting k-uniform family?

The First Construction

Observe that any d edges of $K_{d+1}^{(d)}$ intersect.

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Note that $|\mathcal{A}(k,d)| = (d+1)\binom{n-d-1}{k-d} + \binom{n-d-1}{k-d-1} \sim (d+1)\binom{n}{k-d}$.

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Let $d \leq k$, then the following is a d-wise intersecting family:

$$\mathcal{HM}(k,d) = \{[k+1] \setminus \{i\} : i \in [d-1]\} \cup \{A \in \binom{[n]}{k} : [d-1] \subset A, A \cap [d,k+1] \neq \emptyset\}$$

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Note that

$$|\mathcal{HM}(k,d)| = \binom{n-d+1}{k-d+1} - \binom{n-k-1}{k-d+1} + d - 1 \sim (k-d+2)\binom{n}{k-d}.$$

Our Main Theorem

Conjecture (Hilton-Milner, 1967)

For n sufficiently large, if $\mathcal{F} \subset \binom{[n]}{k}$ *is a nontrivial d-wise intersecting family, then* $|\mathcal{F}| \le \max\{|\mathcal{A}(k,d)|, |\mathcal{HM}(k,d)|\}$.

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Theorem (O-Verstraete, 2019+)

Let k, d *be integers with* $2 \leq d < k$. *For* $n \geq n_0(k,d)$, *if* $\mathcal{F} \subset \binom{[n]}{k}$ *is a nontrivial d-wise intersecting family, then*

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where we may take $n_0(k,d) = d + e(k^2 2^k)^{2^k}(k-d)$.

A Stability Version of Our Main Theorem

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Theorem (O-Verstraete, 2019+)

Let k, d be integers with $2 \leq d < k$. Then for $n_0(k,d) = d + e(k^2 2^k)^{2^k}(k-d)$ and $n > n_0(k,d)$ we have that:

If $2d+1 \geq k$ and $\mathcal{F}$ is a non-trivial d-wise intersecting family with $|\mathcal{F}| > |\mathcal{HM}(k,d)|$, then $\mathcal{F} \subseteq \mathcal{A}(k,d)$.

The Delta System Method

Definition

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Definition

Let $\mathcal{F} \subset \binom{[n]}{k}$ and $X \subset [n]$, then the **core degree** of X in $\mathcal{F}$ is

$$d^{\star}_{\mathcal{F}}(X) := \max\{s : \exists\ \Delta_{k,s} \text{ so that } \text{core}(\Delta_{k,s}) = X\}.$$

The Structure of d-sets with large core degree

Definition

Given a family $\mathcal{F}$, we say $D \in \binom{[n]}{d}$ has **large core degree** if $d^{\star}_{\mathcal{F}}(D) \geq k$. Let $\mathcal{S}_d(\mathcal{F})$ be the collection of such d-sets in $\mathcal{F}$.

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Example

For $n \geq k(k-d)+d$ we have:

$$\mathcal{S}_d(\mathcal{HM}(k,d)) = \{A \in \binom{[k+1]}{d} : [d-1] \subset A\}$$
$$\mathcal{S}_d(\mathcal{A}(k,d)) = K^{(d)}_{d+1}.$$

Sketch of Proof

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If $\mathcal{S} \subset \binom{[k+1]}{d}$ is $(d-1)$-intersecting, then $\mathcal{S}$ is isomorphic to a subfamily of $\mathcal{S}_d(\mathcal{A}(k,d))$ or $\mathcal{S}_d(\mathcal{HM}(k,d))$.

Sketch of Proof cont.

Lemma

If $|\mathcal{S}_d(\mathcal{F})| \geq 3$ *and* $\mathcal{S}_d(\mathcal{F}) \subset \mathcal{S}_d(\mathcal{A}(k,d))$*, then* $\mathcal{F} \subset \mathcal{A}(k,d)$*.*

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Lemma

If $|\mathcal{S}_d(\mathcal{F})| \geq k - d + 1$ *and* $\mathcal{S}_d(\mathcal{F}) \subset \mathcal{S}_d(\mathcal{HM}(k,d))$*, then* $\mathcal{F} \subset \mathcal{HM}(k,d)$.

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We iteratively apply Füredi's Intersection Semilattice lemma to get enough d-sets with large core degree.

Open Problems

Conjecture (O-Verstraete)

For $k > d \geq 2$ and $n \geq kd/(d-1)$, the unique extremal non-trivial d-wise intersecting families of k-element subsets of $[n]$ are $\mathcal{HM}(k,d)$ and $\mathcal{A}(k,d)$.

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Question (O-Verstraete)

Does there exist a degree version of our theorem for $n \geq n_1(k,d)$?

Thanks

Thank you for listening!