Bollobás Set k-tuples

Jason O'Neill
Joint work with Jacques Verstraete

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Talk Overview

In this talk, we will discuss the following topics:

1. Bollobás set pairs
2. Erdős, Goodman and Pósa Correspondence
3. Bollobás set triples
4. Bollobás set k-tuples
5. Biclique covering numbers of Hypergraphs

Antichains in Boolean Lattice

Definition

Given a set family $\mathcal{A} = \{A_1, A_2, \ldots, A_m\} \subseteq 2^{[n]}$, we say $\mathcal{A}$ is an **antichain** if for all $i \neq j \in [m]$, we have that $A_i \subsetneq A_j$.

Given $n \in \mathbb{N}$, how large can an antichain $\mathcal{A} \subseteq 2^{[n]}$ be?

Theorem (Sperner)

If $\mathcal{A} \subseteq 2^{[n]}$ is an antichain, then we have that $|\mathcal{A}| \leq \binom{n}{\lceil \frac{n}{2} \rceil}$

Bollobás set pairs

Definition

Let $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ be families of finite sets, such that $A_i \cap B_j \neq \emptyset$ if and only if $i, j \in [m]$ are distinct, then we say that the pair $(\mathcal{A}, \mathcal{B})$ is a **Bollobás set pair** of size m and write $|(\mathcal{A}, \mathcal{B})| = m$.

Given an antichain $\mathcal{A} = \{A_1, \ldots, A_m\}$, and by letting $B_i := A_i^C$, we see that $\mathcal{B} := \{B_1, \ldots, B_m\}$ is so that $(\mathcal{A}, \mathcal{B})$ is a Bollobás set pair.

Fixing the ground set $[n]$, one can also ask what is

$$\beta_{2,2}(n) := \max\{|(\mathcal{A}, \mathcal{B})| : (\mathcal{A}, \mathcal{B}) \text{ Bollobás set pair}\}.$$

Bollobás set pairs Inequality

Theorem (Bollobás-Lubell-Meshalkin-Yamamoto)

Let $(\mathcal{A}, \mathcal{B})$ be a Bollobás set pair of size m, then

$$\sum_{i=1}^{m} \binom{|A_i \cup B_i|}{|A_i|}^{-1} \leq 1. \tag{1}$$

We have equality in Equation (1), by taking $\mathcal{A} = [n]^{(k)} := \{A \subseteq [n] : |A| = k\}$ and letting $\mathcal{B} = [n]^{(n-k)}$ be the corresponding compliments.

Equation (1) also yields that $\beta_{2,2}(n) = \binom{n}{\lceil \frac{n}{2} \rceil}$.

Lovász Geometric Analog

Theorem (Lovász)

Let $A_1, \ldots, A_m$ be r-dimensional subspaces and $B_1, \ldots, B_m$ be be t-dimensionals subspaces of a vector space V so that $\dim(A_i \cap B_j) = 0 \iff i = j$. *Then* $m \leq \binom{r+t}{t}$

Very rough sketch of proof.

After some geometric reductions involving projections, we may assume the rank of the V is $r + t$. Now, the r-dimensional subspace A_i have a corresponding r vector $\hat{A}_i$ in the exterior power $\Lambda^r(V)$ and the above conditions combined with the wedge product in the exterior algebra shows that $\{\hat{A}_i\}$ are linearly independent. □

Erdős, Goodman and Pósa (EGP) Correspondence

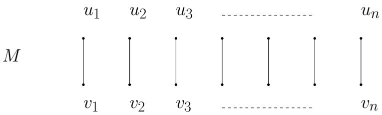
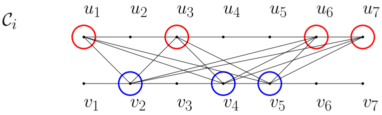


Figure: Consider the bipartite graph $K_{n,n} \setminus M$ with parts $U = \{u_1, u_2, \ldots u_n\}$ and $V = \{v_1, v_2, \ldots, v_n\}$ and consider covering this graph with complete bipartite graphs.

We have that $K_{n,n} \setminus M = \{(u_i, v_j) : i \neq j\}$. Suppose we have that $\{\mathcal{C}_i\}_{i \in [m]}$ is such a covering of $K_{n,n} \setminus M$.

EGP Correspondence Cont.



$\updownarrow$

$i \in A_1, A_3, A_6, A_7$

$i \in B_2, B_4, B_5$

Figure: Given our covering $\{\mathcal{C}_i\}_{i\in[m]}$, we may form $(\mathcal{A}, \mathcal{B})$ in the following manner. Equivalently, we have that $A_i = \{j : \mathcal{C}_j \text{ contains } u_i\}$ and $B_i = \{j : \mathcal{C}_j \text{ contains } v_i\}$.

Claim

$(\mathcal{A}, \mathcal{B})$ is a Bollobás set pair of size n on ground set $[m]$.

EGP Correspondence Cont.

Proof.

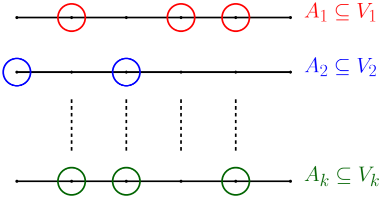
We have that $A_i \cap B_i = \emptyset$ since otherwise there exists a cover $\mathcal{C}_j$ which covers u_i and v_i but $(u_i, v_i) \in M$. We have that for $i \neq j$ that $A_i \cap B_j \neq \emptyset$ since we necessarily cover the edge (u_i, v_j) with at least one of our covers $\mathcal{C}_j$. □

Letting $bc(G)$ be the number of bipartite graphs need to cover G,

$$bc(K_{n,n} \setminus M) = \min\{m : \beta_{2,2}(m) \geq n\} = \min\{m : \binom{m}{\lceil \frac{m}{2} \rceil} \geq n\}.$$

Orlin used EGP correspondence to determine the similarly defined clique covering number of $K_{n,n} \setminus M$.

Biclique covering number



k-partite k-uniform hypergraph $\mathcal{C}$

Figure: Consider a k-uniform k-partite hypergraphs $\mathcal{C}$ where $E(\mathcal{C}) := \{x_{1,i_1}x_{2,i_2}\cdots x_{k,i_k} : i_1 \in A_1, i_2 \in A_i, \ldots, i_k \in A_k\}$

Given a k-uniform hypergraph H, define

$$\mathrm{bc}(H) := \min\{m : \bigcup_{i=1}^{m} E(\mathcal{C}_i) = H\}.$$

Biclique covering number Cont.

Consider the k-uniform complete hypergraph K_n^k. When $k = 2$, we have that $bc(K_n^2) = \log_2(n)$, but for general k this is more challenging.

Körner and Marston show using the powerful notion of *hypergraph entropy* that $bc(K_n^k) \geq (\log \frac{n}{k-1})/(\log \frac{k}{k-1})$.

However, for $n \geq 3$, the limiting value of $bc(K_n^k)/\log n$ as $n \to \infty$ is not known.

Bollobás set triples

Consider three set families $\mathcal{A} = \{A_1, \ldots, A_m\}$, $\mathcal{B} = \{B_1, \ldots, B_m\}$, and $\mathcal{C} = \{C_1, \ldots, C_m\}$. Generalizing to Bollobás set triples, it seems pretty clear that we should have $A_i \cap B_i \cap C_i = \emptyset$ and $A_i \cap B_j \cap C_k \neq \emptyset$ where $|\{i, j, k\}| = 3$.

However, it is unclear what we should impose on $A_i \cap B_i \cap C_j$ or in general $A_{i_1} \cap B_{i_2} \cap C_{i_3}$ with $|\{i_1, i_2, i_3\}| = 2$.

We will first consider Bollobás set triples of threshold $t = 2$ so that $A_{i_1} \cap B_{i_2} \cap C_{i_3} \neq \emptyset \iff |\{i_1, i_2, i_3\}| \geq 2$.

Bollobás set triples of Threshold $t = 2$

Theorem (O-Verstraete)

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be as above, then

$$\sum_{i=1}^{m} \binom{|A_i \cap C_i| + |B_i|}{|B_i|}^{-1} \leq 1. \tag{2}$$

This follows by letting $\mathcal{D} := \{D_i = A_i \cap C_i\}$ and noting that $(\mathcal{B}, \mathcal{D})$ is a Bollobás set pair.

Fixing the ground set $[n]$, one can also ask what is

$$\beta_{3,2}(n) := \max\{|(\mathcal{A}, \mathcal{B}, \mathcal{C})| : (\mathcal{A}, \mathcal{B}, \mathcal{C}) \text{ threshold } t = 2\}?$$

Erdős, Goodman and Pósa Correspondence

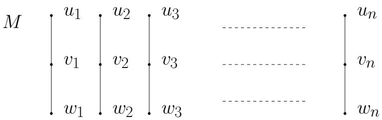


Figure: Consider the 3-partite 3-uniform hypergraph $K_{n,n,n} \setminus M$ with parts $U = \{u_1, u_2, \dots u_n\}$, $V = \{v_1, v_2, \dots, v_n\}$, and $W = \{w_1, w_2, \dots, w_n\}$ and consider covering this graph with complete 3-partite 3-uniform hypergraphs.

We have that $K_{n,n,n} \setminus M = \{(u_i, v_j, w_k) : |\{i, j, k\}| \geq 2\}$. Suppose we have that $\{\mathcal{C}_i\}_{i\in[m]}$ is such a covering of $K_{n,n,n} \setminus M$.

EGP in $(3, 2)$ Setting

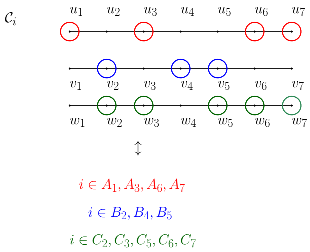


Figure: We have a correspondence between biclique covers of $K_{n,n,n} \setminus M$ and Bollobás set triples of threshold $t = 2$.

As in the case where $k = t = 2$, we have that

$$\nu_{3,2}(n) = bc(K_{n,n,n} \setminus M) = \min\{m : \beta_{3,2}(m) \geq n\}.$$

Bollobás Set triples of threshold $t = 3$

Theorem (O-Verstraete)

Let $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$, $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ *and* $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ *be families of finite sets such that* $A_i \cap B_j \cap C_k \neq \emptyset$ *if and only if* $i, j, k \in [m]$ *are all distinct. Then*

$$\sum_{i=1}^{m} \sum_{\substack{j=1 \\ j \neq i}}^{m} \binom{|A_i \cup B_j \cup C_i|}{|A_i|\ ,\ |B_j \setminus A_i|}^{-1} \leq 1. \tag{3}$$

Fixing the ground set $[n]$, one can also ask what is

$$\beta_{3,3}(n) := \max\{|(\mathcal{A}, \mathcal{B}, \mathcal{C})| : (\mathcal{A}, \mathcal{B}, \mathcal{C}) \text{ threshold } t = 3\}?$$

Erdős, Goodman and Pósa Correspondence

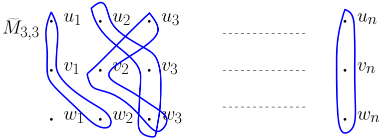


Figure: Consider the 3-partite 3 uniform hypergraph $K_{n,n,n} \setminus \widetilde{M}_{3,3}$ with parts $U = \{u_1, u_2, \ldots u_n\}$, $V = \{v_1, v_2, \ldots, v_n\}$, and $W = \{w_1, w_2, \ldots, w_n\}$ and consider covering this graph with complete 3-partite 3-uniform hypergraphs.

We have that $K_{n,n,n} \setminus \widetilde{M}_{3,3} = \{(u_i, v_j, w_k) : |\{i,j,k\}| = 3\}$. Suppose we have that $\{\mathcal{C}_i\}_{i \in [m]}$ is such a covering of $K_{n,n,n} \setminus \widetilde{M}_{3,3}$.

EGP in $(3,3)$ Setting

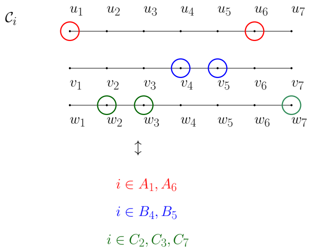


Figure: We have a correspondence between biclique covers of $K_{n,n,n} \setminus \widetilde{M}_{3,3}$ and Bollobás set triples of threshold $t = 3$.

As in the case where $k = t = 2$, and $k = 3$ with $t = 2$ we have

$$\nu_{3,3}(n) = bc(K_{n,n,n} \setminus \widetilde{M}_{3,3}) = \min\{m : \beta_{3,3}(m) \geq n\}.$$

Bollobás Set k-tuples of threshold t

We will consider k-tuples consisting of families $\mathcal{A}_j : 1 \leq j \leq k$ of finite sets with a condition on when the k-wise intersections are nonempty. For integers $k \geq t \geq 2$, we say a **Bollobás set k-tuple with threshold** t is a sequence $(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k)$ of families of sets where $\mathcal{A}_j = \{A_{j,i} : 1 \leq i \leq m\}$ where

$$\bigcap_{j=1}^{k} A_{j,i_j} \neq \emptyset \quad \text{if and only if} \quad |\{i_1, i_2, \ldots, i_k\}| \geq t.$$

When $k = t = 2$, we have precisely a Bollobás set pair. The quantity m is called the *size* of the Bollobás set k-tuple.

Bollobás Set k-tuples Inequality

We are able to prove a Bollobás type inequality for a Bollobás set k-tuple with threshold t.

Consider the specific case where $k = 5$ and $t = 3$ and let $\mathcal{A}^{(1)} = \{A_i^{(1)}\}_{i \in [m]}$, and $\mathcal{A}^{(2)}, \dots, \mathcal{A}^{(5)}$ be defined similarly so that $(\mathcal{A}^{(1)}, \mathcal{A}^{(2)}, \dots, \mathcal{A}^{(5)})$ is a Bollobás set 5-tuple with threshold $t = 3$.

Recall, the Bollobás type Inequality for 3-tuples of threshold $t = 3$:

$$\sum_{i=1}^{m} \sum_{\substack{j=1 \\ j \neq i}}^{m} \binom{|A_i \cup B_j \cup C_i|}{|A_i| \ , \ |B_j \setminus A_i|} \leq 1.$$

Bollobás Set k-tuples Inequality Cont.

Fix a surjective map $\phi : [5] \to [3]$. Say $\phi(1) = \phi(3) = 1$, $\phi(2) = \phi(5) = 2$ and $\phi(4) = 3$.

Define $\mathcal{D}(\phi)^{(1)} = \{D_i(\phi)^{(1)} := A_i^{(1)} \cap A_i^{(3)}\}_{i \in [m]}$
$\mathcal{D}(\phi)^{(2)} = \{D_i(\phi)^{(2)} := A_i^{(2)} \cap A_i^{(5)}\}_{i \in [m]}$
$\mathcal{D}(\phi)^{(3)} = \{D_i(\phi)^{(3)} := A_i^{(4)}\}_{i \in [m]}$

Claim

$(\mathcal{D}(\phi)^{(1)}, \mathcal{D}(\phi)^{(2)}, \mathcal{D}(\phi)^{(3)})$ *is a Bollobás set 3-tuple of threshold* $t = 3$ *and hence we have that*

$$\sum_{i=1}^{m} \sum_{\substack{j=1 \\ j \neq i}}^{m} \begin{pmatrix} |D_i(\phi)^{(1)} \cup D_j(\phi)^{(2)} \cup D_i(\phi)^{(3)}| \\ |D_i(\phi)^{(1)}| \,,\ |D_i(\phi)^{(2)} \setminus D_i(\phi)^{(1)}| \end{pmatrix} \leq 1.$$

Bollobás Set k-tuples Inequality Cont.

Observe that we can do this for all $\phi : [5] \to [3]$ where ϕ is a surjection, and to this end let

$$S(\phi, 5, 3, m) := \sum_{i=1}^{m} \sum_{\substack{j=1 \\ j \neq i}}^{m} \binom{|D_i(\phi)^{(1)} \cup D_j(\phi)^{(2)} \cup D_i(\phi)^{(3)}|}{|D_i(\phi)^{(1)}| \, , \, |D_i(\phi)^{(2)} \setminus D_i(\phi)^{(1)}|}.$$

Theorem

Let $(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_5)$ be a Bollobás set 5-tuple of thereshold $t = 3$ and let $\Phi := \{\phi : [5] \to [3]$ surjection $\}$, then we have that

$$\max_{\phi \in \Phi} S(\phi, 5, 3, m) \leq 1.$$

EGP in (k, t) Setting

Consider the complete k-partite k-uniform hypergraph with parts $X_i = \{x_{i1}, x_{i2}, \ldots, x_{in}\}$ for $1 \leq i \leq k$. Then, we let $\widetilde{M} := \{x_{1i_1} \ldots x_{ki_k} : |\{i_1, \ldots, i_k\}| < t\}$ and consider the hypergraph

$$H(k, t, n) := K_{n,n\ldots,n} \setminus \widetilde{M}.$$

Hence we have that $e = (x_{1,i_1}, \ldots, x_{k,i_k})$ is so that

$$e \in H(k, t, n) \iff |\{i_1, \ldots, i_k\}| \geq t.$$

Using the Erdős-Goodman-Pósa Correspondence, we have

$\{$Bollobás set k-tuples of threshold $t\} \leftrightarrow \{$biclique covers of H(k,t,n)$\}$

EGP in (k, t) Setting Cont.

Given $t \leq k \in \mathbb{N}$, let $\beta_{k,t}(m)$ be the largest Bollobás set k-tuple with threshold t on ground set $[m]$, then we have that

$$\nu_{k,t}(n) = bc(H(k,t,n)) = \min\{m : \beta_{k,t}(m) \geq n\}. \quad (4)$$

Equation (4) yields that a probabilistic construction of a Bollobás set k-tuple of threshold t and hence a lower bound of $\beta_{k,t}(m)$ yields an *upper bound* on the biclique covering number $bc(H(k,t,n))$.

We can get a *lower bound* on $bc(H(k,t,n))$ through a variety of different techniques.

Probabilistic construction when $t = 2$

Recall in the case where the threshold $t = 2$ we are considering the k-partite, k-regular hypergraph where we remove a matching which we denote as $H(k, 2, n)$. We have

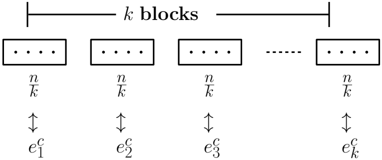


Figure: Observe that $|e_i| = \frac{n(k-1)}{k}$ and that $\bigcap_i e_i = \emptyset$

Let $f_1, \ldots, f_x$ be random bijections from $\bigcup_{i=1}^{k} e_i \to [n]$ and let

$$A_{i,j} = f_j(e_i) \text{ and } \mathcal{A}_i = \{A_{i,j}\}_{j \in [x]}.$$

Probabilistic Construction when $t = 2$ Cont.

We have that $A_{1,j} \cap A_{2,j} \cap \cdots \cap A_{k,j} = \emptyset$ for all $j \in [x]$. We then compute the expected number of k-tuples $\{i_1, \ldots, i_k\}$ so that $|\{i_1, \ldots, i_k\}| \geq 2$ and $A_{1,i_1} \cap A_{2,i_2} \cap \cdots \cap A_{k,i_k} = \emptyset$ and show this is small for suitable x.

Theorem (O-Verstraete)

For $n \geq k \geq 2$, we have that

$$\frac{k}{\log(ke)} \leq \frac{bc(H(k,2,n))}{\log n} \leq \frac{k-1}{-\log(1-e^{-1})}.$$

Probabilistic Construction when $t = k$

We have the following probabilistic construction of a Bollobás set k-tuple of threshold $t = k$. Consider random and uniform colorings $f_1, \ldots, f_N$ where $f_i : [n] \to [k]$ and define $A_{l,i} = f_i^{-1}(l)$ and $\mathcal{A}_l = \{A_{l,i}\}_{i \in [N]}$.

We have that whenever $|\{i_1, i_2, \ldots, i_k\}| < k$, that $A_{1,i_1} \cap A_{2,i_2} \cap \cdots \cap A_{k,i_k} = \emptyset$. Letting X be the number of k-tuples $\{i_1, \ldots, i_k\}$ with disjoint entries whose k-wise intersection is empty;

$$\mathbf{E}[X] \leq N^k (1 - \frac{1}{k^k})^n < \frac{N}{2}$$

provided that $N < \left(\frac{k^k}{k^k - 1}\right)^{\frac{n}{k-1}}$.

Probabilistic Construction when $t = k$ Cont.

Hence, we have the lower bound $\beta_{k,k}(n) \geq \left(\frac{k^k}{k^k - 1}\right)^{\frac{n}{k-1}}$ which yields an upper bound on $\mathrm{bc}(H(k,k,n))$.

The lower bound follows from a more involved double-counting argument. We thus have that:

Theorem (O-Verstraete)

For $k \geq 2$, if we take $n \geq k^3$, then

$$\frac{1}{3k}(k-1)^{k-1} \leq \frac{bc(H(k,2,n))}{\log(n)} \leq \frac{2}{\log(e)} k^{k+1}.$$

Lower Bound on $\mathrm{bc}(H(k,t,n))$ when $2 < t < k$

We will consider the case where $k = 6$ and $t = 4$ below.

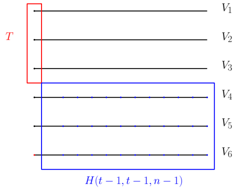


Figure: Given a subset $T \subset [k]$ so that $|T| = k - t + 1$, we may consider the hypergraph $H_T(1) \subset H(k,t,n)$ where we force all indices in T to have vertex 1.

By considering the fixed $k - t + 1$ elements for a given T, we have a link $(t-1)$-uniform hypergraph which is naturally isomorphic to $H(t-1, t-1, n-1)$.

Lower Bound on $\mathrm{bc}(H(k,t,n))$ when $2 < t < k$ Cont.

We therefore need at least $bc(H(t-1,t-1,n-1))$ bicliques in a biclique cover to cover edges in $H_T(1)$.

Distinct subsets $T, T' \in [k]^{(k-t+1)}$ cannot be covered by the same biclique, which yields that

$$\binom{k}{k-t+1}\mathrm{bc}(H(t-1,t-1,n-1)) \leq \mathrm{bc}(H(k,t,n)).$$

Using the bound on $\mathrm{bc}(H(t-1,t-1,n-1)$, we get

$$\binom{k}{t-1}\frac{(t-2)^{t-2}}{3(t-1)} \leq \frac{\mathrm{bc}(H(k,t,n)}{\log(n)}$$

Result when $2 < t < k$

An involved probabilistic construction of a biclique cover $\mathcal{C}$ where

$$|\mathcal{C}| \leq \binom{k}{t-1} \frac{t+1}{t} t^t \log(n)$$

yields the upper bound and hence we have that

Theorem (O-Verstraete)

For $k \geq 2$, if we take $n \geq k^3$, then

$$\binom{k}{t-1} \frac{(t-2)^{t-2}}{3(t-1)} \leq \frac{bc(H(k,t,n))}{\log(n)} \leq \binom{k}{t-1} \frac{t+1}{t} t^t.$$

Relating back to $\beta_{k,t}(n)$

Using $bc(H(k,t,n)) = \min\{m : \beta_{k,t}(m) \geq n\}$, we have that

Theorem (O-Verstraete)

$$\frac{1-\log(e-1)}{k-1} \leq \frac{\log \beta_{k,2}(n)}{n} \leq \frac{\log(ke)}{k}.$$

$$\frac{1}{(k-1)k^k} \leq \frac{\log \beta_{k,k}(n)}{n} \leq \frac{3k}{(k-1)^{k-1}} .$$

$$\leq \frac{\log \beta_{k,t}(n)}{n} \leq .$$

Open Problems

1. Improve the bounds on $\text{bc}(H(k,t,n))$.
2. Find an explicit construction of an exponential size Bollobás set 3-tuple of threshold 3.
3. Show that the Bollobás type Inequality is tight in any case where $t \geq 3$.
4. Compute $\lim_{n\to\infty} \frac{\text{bc}(H(3,2,n))}{\log(n)}$ if it exists

Thank you for listening!