Building new posets from old: The Tesler poset

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Motivating Example

Let U_3 be the set of 3×3 upper-triangular matrices with non-negative integer entries.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

We say $A \in U_3$ is a **Tesler matrix** if the **hook sums** are so that $a_{13} + a_{12} + a_{11} = 1$, $a_{23} + a_{22} - a_{12} = 1$ and $a_{33} - a_{23} - a_{13} = 1$.

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How many 3×3 Tesler matrices are there?

Motivating Example Cont.

$$\begin{bmatrix} 0 & 1 & 0 \\ & 0 & 2 \\ & & 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ & 0 & 1 \\ & & 3 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ & 1 & 1 \\ & & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ & 0 & 1 \\ & & 2 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 \\ & 1 & 0 \\ & & 2 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ & 2 & 0 \\ & & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{bmatrix}$$

Figure: Here they are. There are seven so we write $T(1^3) = 7$.

Formal Definitions

Let U_n be the set of $n \times n$ upper-triangular matrices with non-negative integer entries. Given $A \in U_n$, where $A = (a_{i,j})$, we define the **hook sum** h_k for $1 \leq k \leq n$ to be the sum of all entries weakly right of $a_{k,k}$ minus all entries strictly above it. That is,

$$h_k := (a_{k,k} + a_{k,k+1} + \cdots + a_{k,n}) - (a_{1,k} + a_{2,k} + \cdots + a_{k-1,k})$$

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We define the **hook sum vector** as the n-dimensional vector $(h_1, \ldots, h_n)$. A **Tesler matrix** $A \in U_n$ is such that $h_k = 1$ for all $1 \leq k \leq n$. As before, we let $T(1^n)$ be the number of such matrices.

Tesler Matrix viewed as Digraph

$$\begin{pmatrix} 0 & 1 & 0 \\ & 1 & 1 \\ & & 2 \end{pmatrix} \longleftrightarrow$$

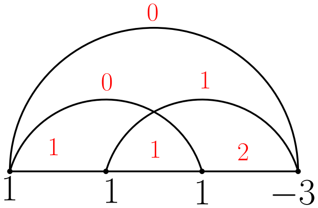

Figure: A Tesler matrix of size n can also be represented as an integral flow on the complete directed graph on $(n+1)$ vertices with net flows equal to $(1, 1, \ldots, 1, -n)$. The idea is that a vertex has inflows or positive flow which corresponds to adding the terms in the row and negative flow or outflows corresponding to subtracting along the column.

Why we care

Space of Diagonal Harmonics DH_n is a bigraded S_n-module:

$$DH_n = \left\{ f \in \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots y_n] : \sum_{i=1}^{n} \frac{\delta^k}{\delta x_i} \frac{\delta^l}{\delta y_i} f = 0 \, \forall \, k+l > 0 \right\}$$

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We then have that the bigraded Hilbert series of DH_n is so that

$$Hilb(DH_n; q, t) := \sum_{i,j=0}^{n} \dim(DH_n)_{i,j} q^i t^j = \sum_{A} \mathrm{wt}_{q,t}(A)$$

where the sum is over Tesler matrices A.

What's so special about 1?

We will also consider these hook sums to take on other values besides 1. We denote the number of matrices in U_n with a hook sum vector of $(h_1, \ldots, h_n)$ as $T(h_1, \ldots, h_n)$ and refer to these as **generalized Tesler matrices**.

We will pay particular attention to the case where $h_i \in \{0, 1\}$, but there are some nice results when $h_i = i$

Bounds on $T(h_1, \ldots, h_n)$

Theorem (Zeilberger)

$$T(1, 2, \ldots, n) = \prod_{i=1}^{n} C_i$$

where $C_i := \frac{1}{i+1}\binom{2i}{i}$ *is the ith Catalan number.*

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Proposition (Mészáros, Morales, Rhoades)

$$n! \leq T(1^n) \leq 2^{\binom{n}{2}}$$

Minor Improvements

Proposition (O'N.)

$$\prod_{i=1}^{n-1}(2i-1) \leq T(1^n) \leq 2^{\binom{n-2}{2}-1} \cdot 3^n$$

This is still not a very tight bound, and in hopes to improve this, we need more powerful machinery and one potential method is realized when consider the **Tesler poset** on Tesler matrices and its characteristic polynomial.

Poset definitions

Given a finite, ranked poset P, we can recursively define the Möbius function:

$$\mu(s,u) = \left\{ \begin{array}{ll} 1, & \text{for } s = \mu \\ -\sum_{s \leq t < u} \mu(s,t) & \text{for } s < u \end{array} \right\}$$

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We now define the characteristic polynomial of a ranked poset P.

$$\chi(P;q) = \sum_{A \in P} \mu(\hat{0}, A) q^{\rho(P) - \rho(A)}$$

where $\rho(A)$ is the rank of A in P and $\rho(P)$ is the rank of the poset.

Boolean Lattice B_n

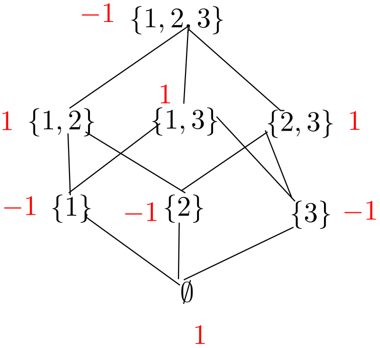

Figure: The Boolean lattice with the values of the Möbius function in red. We can check that $\chi(B_3; q) = q^3 - 3 \cdot q^2 + 3 \cdot q^1 - 1 = (q-1)^3$ and in general we have that $\chi(B_n; q) = (q-1)^n$

Cover Relation for Tesler poset

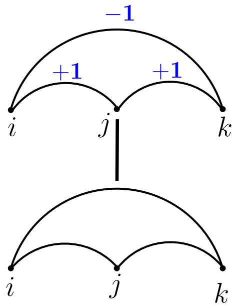

Figure: We can also translate the cover relation definition in terms of matrices as well, but this is the most concise definition.

We can now look at the Tesler poset for the case where $n = 3$.

Tesler poset for $n = 3$

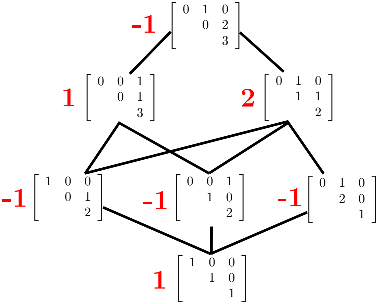

Figure: The Tesler poset $P(1^3)$ with the values of the Möbius function in red. We can check that $\chi(P(1^3); q) = q^3 - 3q^2 + 3 - 1 = (q-1)^3$

Armstrong's conjecture

Conjecture (Armstrong)

$\chi(P(1^n); q) = (q-1)^{\binom{n}{2}}$

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Theorem (O'N)

Let $\alpha = (\alpha_{n-1}, \ldots, \alpha_0) \in \{0,1\}^n$ and $P(\alpha)$ be the poset on generalized Tesler matrices $\mathcal{T}(\alpha)$. Then, letting $w(\alpha) = \sum_{i=0}^{n-1} i \cdot \alpha_i$, we have that

$$\chi(P(\alpha); q) = (q-1)^{w(\alpha)}$$

Proof of Armstrong's conjecture

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Proof.

$$w(1, 1, \cdots, 1) = \sum_{i=0}^{n-1} i \cdot \alpha_i = \sum_{i=0}^{n-1} i = \binom{n}{2}$$

□

Hallam and Sagan method

Consider ranked posets $P_1, \ldots, P_k$ for which the characteristic polynomial is known, and to consider $Q = P_1 \times \cdots \times P_k$. We recall the following facts regarding the characteristic polynomial of posets.

1) If $P \cong P'$, then $\chi(P; q) = \chi(P'; q)$
2) $\chi(P_1 \times P_2; q) = \chi(P_1; q) \cdot \chi(P_2; q)$

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Then, they define an equivalence relation $(\sim)$ to identify elements in Q such that $Q/\sim \cong P$. The process of identifying elements leaves the characteristic polynomial unchanged if the equivalence relation satisfies certain conditions.

Hallam and Sagan method cont.

Lemma (Hallam and Sagan)

Let Q be a ranked poset as above and $\sim$ be an equivalence relation on Q which is homogeneous, preserves rank and satisfies the summation condition. Then

$$\chi(P;q) = \chi(Q/\sim;q) = \chi(Q;q)$$

Applying Method

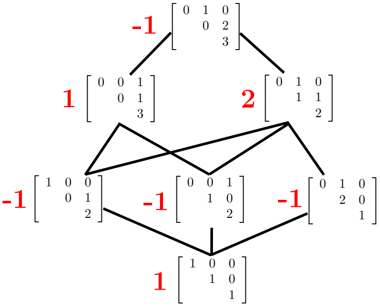

Figure: It's not quite clear how to define an equivalence relation due to symmetry of Boolean lattice and lack of symmetry of Tesler poset.

Jordan Form

The Tesler poset $P(1, 0, \ldots, 0)$

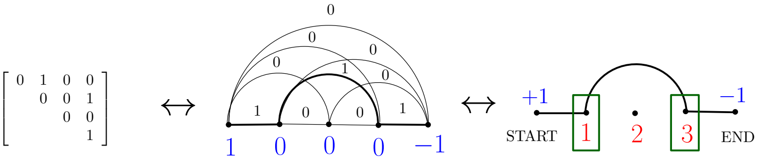

Figure: We have that $P(1, 0, \ldots, 0) \cong B_{n-1} \cong P(1, 0, \ldots, 0, 1)$

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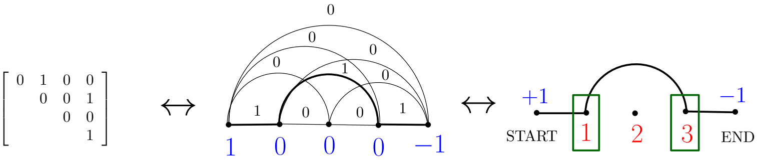

Figure: We have that $P(1, 0, \dots, 0) \cong B_{n-1} \cong P(1, 0, \dots, 0, 1)$

This gives us a way of relating our building blocks of the Boolean lattice in terms of Tesler matrices with a simple hook sum.

Equivalence relation on the product Poset

$$\begin{bmatrix} 0 & 1 & 0 \\ & 1 & 0 \\ & & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ & 1 & 1 \\ & & 2 \end{bmatrix}$$

Figure: Note that the map $+ : T(1,0,1) \times T(1,0) \to T(1,1,1)$ is well-defined.

We then have $(A_1, B_1) \sim (A_2, B_2)$ if $A_1 + B_1 = A_2 + B_2$ as matrices.

Example for Tesler poset $P(1^3)$

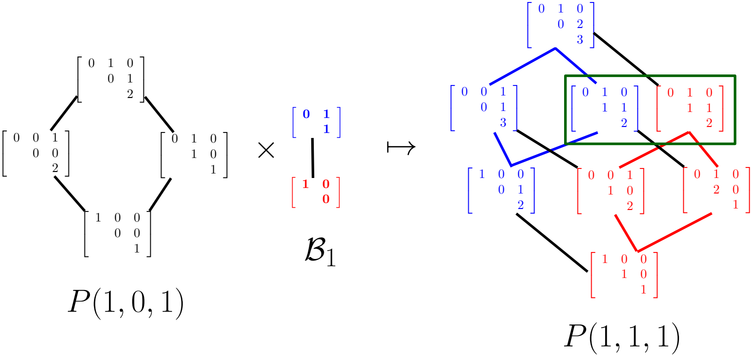

Figure: The Tesler poset $P(1^3)$ can be realized as the product of Boolean lattices of size 2 and 1 after applying an equivalence relation.

Main Result

By inductively applying the Sagan-Hallam product method for the posets, we get:

Theorem (O'N)

Let $\alpha = (\alpha_{n-1}, \ldots, \alpha_0) \in \{0,1\}^n$ *and* $P(\alpha)$ *be the poset on generalized Tesler matrices* $\mathcal{T}(\alpha)$. *Then, letting* $w(\alpha) = \sum_{i=0}^{n-1} i \cdot \alpha_i$, *we have that*

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What about more general $\alpha \notin \{0,1\}^n$?

Characteristic polynomial not always a power of $(q-1)$

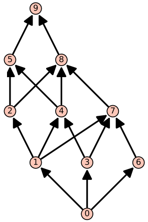

Figure: We have that $\chi(P(1,2,1);q) = q(q-1)^3$

Also, one can compute

$$\chi(P(1,2,3,4);q) = (q-1)^4(q^5 - 2q^4 + 4q^3 - 6q^2 + 3q + 1)$$

Relating Back to the bounds on $T(1^n)$

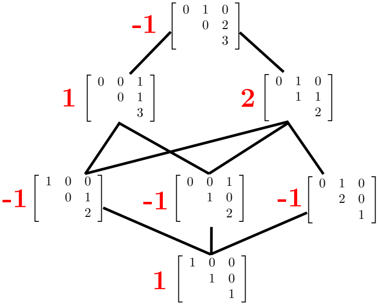

Figure: Note that the element that corresponds to an equivalence class has Möbius value of 2. The value of the Möbius function is the sum of the values of the elements in the equivalence class.

Bound to Improve

Proposition (O'N)

Let $\mu(\cdot)$ be the Möbius function for the Tesler poset $P(1^n)$. If for all $A \in \mathcal{T}(1^n)$ we have that $|\mu(\hat{0}, A)| \leq f(n)$, then we have that:

$$T(1^n) \geq \frac{2^{\binom{n}{2}}}{f(n)}$$

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We note that we have

$$\sum_{A} |\mu(\hat{0}, A)| \geq 2^{\binom{n}{2}}$$

Hence, if for all $A \in \mathcal{T}(1^n)$ we have that $|\mu(\hat{0}, A)| \leq f(n)$, then we would have that $T(1^n) \cdot f(n) \geq 2^{\binom{n}{2}}$.

Bound to Improve cont.

We would find such a bound on the Möbius function for the Tesler poset $P(1^n)$ by analyzing the size of the equivalence classes that we get when we use Hallam-Sagan's method.

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Conjecture (O'N.)

Let $\alpha = (1, 1, \ldots, 1)$ and $P(\alpha)$ be the Tesler poset with Möbius function $\mu(\cdot)$. Then we can have the following lower bound on the Möbius function

$$|\mu(\hat{0}, A)| \leq n!$$

Thank you!

Thank you for your attention! Any questions?