Midterm 1 Solutions

Version A

(2 points) 1. Carefully read the instructions on the front page of this exam.

(10 points) 2. Compute the definite integral

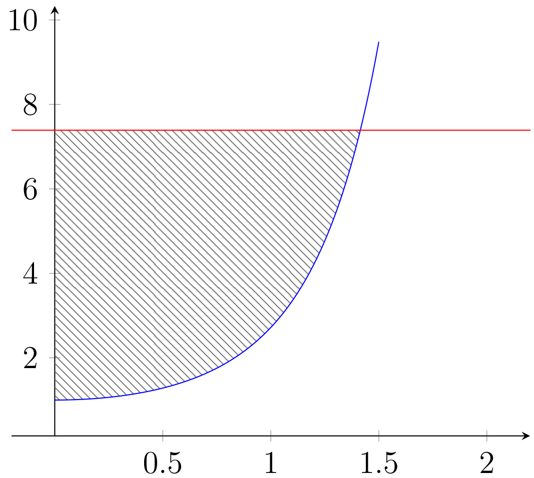
$$\int_{\log(\frac{3\pi}{4})}^{\log(\pi)} \sec^2(e^y)e^y dy$$

Solution: We start with the substitution $u = e^y$, so that $\frac{du}{dy} = e^y$. Then we have that

$$\begin{aligned}
\int_{\log(\frac{3\pi}{4})}^{\log(\pi)} \sec^2(e^y)e^y dy &= \int_{\log(\frac{3\pi}{4})}^{\log(\pi)} \sec^2(u(y))\frac{du}{dy}dy \\
&= \int_{y=\log(\frac{3\pi}{4})}^{y=\log(\pi)} \sec^2(u)du \\
&= \int_{u=\frac{3\pi}{4}}^{u=\pi} \sec^2(u)du \\
&= [\tan(u)]_{\frac{3\pi}{4}}^{\pi} \\
&= \tan(\pi) - \tan(\frac{3\pi}{4}) \\
&= 0 - (-1) \\
&= 1.
\end{aligned}$$

(10 points) 3. Find the volume of the solid obtained by rotating the region bounded by the curves $x = \sqrt{\log(y)}, y = e^2$, and $x = 0$ around the y-axis.

Solution: The graph of the area in question looks like this:



Here is how you could figure out how to set up the integral even without drawing the graph. Since we are rotating around the y-axis—which is the same as the line $x = 0$—we know that we need to be integrating the cross sectional area with respect to y. The curve $x = \sqrt{\log(y)}$ intersects $x = 0$ when $y = 1$, and (since $\log(y)$ is positive for $y > 1$) lies to the right of $x = 0$. This means that in the formula for volume, the outer radius of the cross section is given by the distance between $x = \sqrt{\log(y)}$ and the y-axis, so $r_{outer}(y) = \sqrt{\log(y)}$. The inner radius is then the distance between $x = 0$ and itself, meaning $r_{inner}(y) = 0$. The bounds of integration are from $y = 1$ to $y = e^2$.

Now we have everything we need to set up the integral:

$$\begin{aligned} V &= \pi \int_1^{e^2} r_{outer}(y)^2 - r_{inner}(y)^2 dy \\ &= \pi \int_1^{e^2} (\sqrt{\log(y)})^2 - 0^2 dy \\ &= \pi \int_1^{e^2} \log(y) dy. \end{aligned}$$

To calculate the integral, we can integrate by parts. Set

$$u = \log(y), \frac{dv}{dy} = 1 \Rightarrow dv = dy$$

so that

$$\frac{du}{dy} = \frac{1}{y} \Rightarrow du = \frac{1}{y} dy, v = \int dy = y.$$

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From the formula for integration by parts,

$$
\begin{aligned}
V &= \pi \int_1^{e^2} \log(y) dy \\
&= \pi \left[\int u dv \right]_1^{e^2} \\
&= \pi \left[\int uv - \int v du \right]_1^{e^2} \\
&= \pi \left[y \log(y) - \int \frac{y}{y} dy \right]_1^{e^2} \\
&= \pi \left[y \log(y) - y \right]_1^{e^2} \\
&= \pi \left[(e^2 \log(e^2) - e^2) - (1 \cdot \log(1) - 1) \right] \\
&= \pi \left[(2e^2 - e^2) - (0 - 1) \right] \\
&= \pi(e^2 + 1).
\end{aligned}
$$

(5 points) 4. (a) Compute the antiderivative

$$\int ue^u du$$

Solution: We can solve this one using integration by parts. A good choice is to let $w = u$, and $ds = e^u du$ (we're using w and s instead of u and v to avoid confusion, since u is already taken). Taking the appropriate derivatives and antiderivatives, we end up with

$$w = u, dw = du, ds = e^u du, s = e^u.$$

Now the formula for integration by parts says that

$$\int wds = ws - \int sdw$$

which brings us to

$$\begin{aligned}\int ue^u du &= ue^u - \int e^u du \\ &= ue^u - e^u + C.\end{aligned}$$

(5 points) (b) Compute the antiderivative

$$\int x^{13} e^{x^7} dx$$

Solution: The e^{x^7} looks a little intimidating, so we try to get rid of it using the substitution $u = x^7$. Then $du = 7x^6 dx$, which means that $dx = \frac{1}{7x^6} du$. Our integral then translates to

$$\begin{aligned} \int \frac{x^{13}}{7x^6} e^u du &= \int \frac{1}{7} x^7 e^u du \\ &= \frac{1}{7} \int u e^u du. \end{aligned}$$

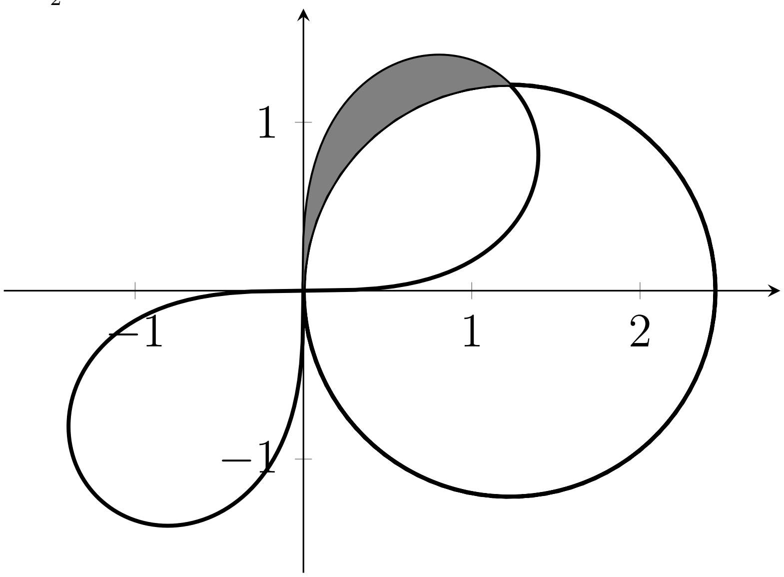
From part a, we know that

$$\frac{1}{7} \int u e^u du = \frac{1}{7}(u e^u - e^u) + C$$

and remembering that $u = x^7$, we arrive at

$$\int x^{13} e^{x^7} dx = \frac{1}{7}(x^7 e^{x^7} - e^{x^7}) + C.$$

(10 points) 5. Find the area of the shaded region between the lemniscate $r^2 = 3\sin(2\theta)$ and the circle $r = \sqrt{6}\cos(\theta)$. Hint: you may find the formulas $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ and $\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$ useful.



Solution: We first find the points of intersection between the two curves, by setting the two functions equal and solving for θ. In this case, we can square the equation of the circle to and set it equal to the equation for the lemniscate:

$$\begin{aligned} 3\sin(2\theta) &= (\sqrt{6}\cos(\theta))^2 \\ &= 6\cos^2(\theta). \end{aligned}$$

The given formula simplies this to

$$\begin{aligned} 6\sin(\theta)\cos(\theta) &= 6\cos^2(\theta) \\ \Rightarrow \sin(\theta)\cos(\theta) &= \cos^2(\theta). \end{aligned}$$

The solutions to this equation are when $\cos(\theta) = 0$ (so both sides are 0), or when $\cos(\theta) \neq 0$ and we can divide both sides by $\cos(\theta)$ to obtain a solution to

$$\cos(\theta) = \sin(\theta)$$

which happens when $\tan(\theta) = 1$. The solutions to $\cos(\theta) = 0$ between 0 and 2π are

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

while the solutions to $\tan(\theta) = 1$ are

$$\theta = \frac{\pi}{4}, \frac{5\pi}{4}.$$

From the graph, we can see that the rightmost intersection point corresponds to $r \neq 0$, so $\cos(\theta) \neq 0$ and we are at the angle $\frac{\pi}{4}$ (because the point of intersection lies in the first quadrant) [1]. The other intersection point occurs when $r = 0$, and the next solution to $\cos(\theta) = 0$ as we travel counterclockwise from $\theta = \frac{\pi}{4}$ is $\theta = \frac{\pi}{2}$. We can also read from the graph that the lemniscate is on the outside in between these two bounds.

Plugging everything in to the formula for the area of the sector enclosed by two polar curves yields

$$\begin{aligned}
A &= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} r_{outer}^2 - r_{inner}^2 d\theta \\
&= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 3\sin(2\theta) - 6\cos^2(\theta) d\theta \\
&= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 3\sin(2\theta) - 6 \cdot \frac{1+\cos(2\theta)}{2} d\theta \\
&= \frac{3}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(2\theta) - 1 - \cos(2\theta) d\theta \\
&= \frac{3}{2}\left[-\frac{1}{2}\cos(2\theta) - \theta - \frac{1}{2}\sin(2\theta)\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
&= \frac{3}{2}\left[(-\frac{1}{2}\cos(2\cdot\frac{\pi}{2}) - \frac{\pi}{2} - \frac{1}{2}\sin(2\cdot\frac{\pi}{2})) - (-\frac{1}{2}\cos(2\cdot\frac{\pi}{4}) - 1\frac{\pi}{4} - \frac{1}{2}\sin(2\cdot\frac{\pi}{4}))\right] \\
&= -\frac{3}{2}\left[(\frac{1}{2}\cos(\pi) + \frac{\pi}{2} + \frac{1}{2}\sin(\pi)) - (\frac{1}{2}\cos(\frac{\pi}{2})) + \frac{\pi}{4} + \frac{1}{2}\sin(\frac{\pi}{2}))\right] \\
&= -\frac{3}{2}\left[(\frac{1}{2}\cdot -1 + \frac{\pi}{2} + \frac{1}{2}\cdot 0) - (\frac{1}{2}\cdot 0 + \frac{\pi}{4} + \frac{1}{2}\cdot 1\right] \\
&= -\frac{3}{2}(\frac{\pi}{4} - 1) \\
&= \frac{3}{2}(1 - \frac{\pi}{4}).
\end{aligned}$$

[1] actually, we could just as easily have started with $\frac{5\pi}{4}$. This is because the points $(\cos(\frac{5\pi}{4}), \frac{5\pi}{4})$ and $(\cos(\frac{\pi}{4}), \frac{\pi}{4})$ are the same, as we see by noticing that $\cos(\frac{5\pi}{4}) = -\cos(\frac{\pi}{4})$. The only difference is that we would need to use a different upper bound later in our integral