(2 points) 1. Carefully read the instructions on the front page of this exam.

(10 points) 2. Compute the antiderivative (Hint: if you use complex exponentials, make sure you end up with real functions in your final answer)

$$\int \sin^2(x)\cos(\pi x)dx$$

Solution: Method 1 (Complex exponentials)
As always, we start by writing sine and cosine in terms of complex exponentials and expand:

$$\begin{aligned}
\int \sin^2(x)\cos(\pi x)dx &= \int (\frac{e^{ix}-e^{-ix}}{2i})^2\frac{e^{\pi ix}+e^{-\pi ix}}{2}dx \\
&= \int \frac{(e^{ix}-e^{-ix})^2}{-4}\frac{e^{\pi ix}+e^{-\pi ix}}{2}dx \\
&= -\frac{1}{8}\int (e^{ix}-e^{-ix})^2(e^{\pi ix}+e^{-\pi ix})dx \\
&= -\frac{1}{8}\int (e^{2ix}+e^{-2ix}-2)(e^{\pi ix}+e^{-\pi ix})dx \\
&= -\frac{1}{8}\int e^{(2+\pi)ix}+e^{(\pi-2)ix}+e^{(2-\pi)ix}+e^{(-2-\pi)ix}-2(e^{\pi ix}+e^{-\pi ix})dx \\
&= -\frac{1}{8}\int e^{(2+\pi)ix}+e^{-(2+\pi)ix}+e^{(\pi-2)ix}+e^{-(\pi-2)ix}-2(e^{\pi ix}+e^{-\pi ix})dx
\end{aligned}$$

Now we have two options: either rewrite these complex exponentials as sine and cosine, then integrate, or integrate, then rewrite as sine and cosine afterward. To save space, we'll try the former:

$$e^{(2+\pi)ix}+e^{-(2+\pi)ix}+e^{(\pi-2)ix}+e^{-(\pi-2)ix}-2(e^{\pi ix}+e^{-\pi ix}) = 2\cos((2+\pi)x)+2\cos((\pi-2)x)-4\cos(\pi x),$$

meaning

$$\begin{aligned}
\int \sin^2(x)\cos(\pi x)dx &= -\frac{1}{8}\int e^{(2+\pi)ix}+e^{-(2+\pi)ix}+e^{(\pi-2)ix}+e^{-(\pi-2)ix}-2(e^{\pi ix}+e^{-\pi ix})dx \\
&= -\frac{1}{4}\int \cos((2+\pi)x)+\cos((\pi-2)x)-2\cos(\pi x)dx \\
&= -\frac{1}{4}[\frac{\sin((2+\pi)x)}{2+\pi}+\frac{\sin((\pi-2)x)}{\pi-2}-\frac{2}{\pi}\sin(\pi x)]+C.
\end{aligned}$$

Method 2 (Integration by parts)
Let $u=\sin^2(x), dv=\cos(\pi x)dx$. By the chain rule, $du = 2\sin(x)\cos(x)dx$, which we know by a trig identity (one we could remember using complex exponentials, for example, or just memorize) is $\sin(2x)$. Since $v=\frac{1}{\pi}\sin(\pi x)$, we know that

$$\int \sin^2(x)\cos(\pi x)dx = \frac{1}{\pi}\sin^2(x)\sin(\pi x) - \frac{1}{\pi}\int \sin(2x)\sin(\pi x)dx.$$

To find this second integral, we have to integrate by parts twice (or use the angle addition formulas, but that is essentially method 1). First let

$$u = \sin(2x), dv = \sin(\pi x) \Rightarrow du = 2\cos(2x)dx, v = -\frac{1}{\pi}\cos(\pi x).$$

Then

$$\int \sin(2x)\sin(\pi x)dx = -\frac{1}{\pi}\sin(2x)\cos(\pi x) + \frac{2}{\pi}\int \cos(2x)\cos(\pi x)dx.$$

Finally, we set

$$u = \cos(2x), dv = \cos(\pi x)dx \Rightarrow du = -2\sin(2x), v = \frac{1}{\pi}\sin(\pi x),$$

resulting in

$$\int \cos(2x)\cos(\pi x)dx = \frac{1}{\pi}\sin(\pi x)\cos(2x) + \frac{2}{\pi}\int \sin(2x)\sin(\pi x).$$

Plugging this into our earlier calculation

$$\int \sin(2x)\frac{1}{\pi}\sin(\pi x)dx = -\frac{1}{\pi}\sin(2x)\cos(\pi x)+\frac{2}{\pi}[\frac{1}{\pi}\sin(\pi x)\cos(2x)+\frac{2}{\pi}\int \sin(2x)\sin(\pi x)dx],$$

hence

$$(1-(\frac{4}{\pi^2})\int \sin(2x)\sin(\pi x)dx = \frac{1}{\pi}(\frac{2}{\pi}\sin(\pi x)\cos(2x) - \sin(2x)\cos(\pi x)),$$

which implies

$$\begin{aligned}\int \sin(2x)\sin(\pi x)dx &= (1-\frac{4}{\pi^2})^{-1}\frac{1}{\pi}(\frac{2}{\pi}\sin(\pi x)\cos(2x) - \sin(2x)\cos(\pi x)) \\ &= (\frac{\pi^2-4}{\pi^2})^{-1}\frac{1}{\pi}(\frac{2}{\pi}\sin(\pi x)\cos(2x) - \sin(2x)\cos(\pi x)) \\ &= \frac{\pi^2}{\pi^2-4}\frac{1}{\pi}(\frac{2}{\pi}\sin(\pi x)\cos(2x) - \sin(2x)\cos(\pi x)) \\ &= \frac{\pi}{\pi^2-4}(\frac{2}{\pi}\sin(\pi x)\cos(2x) - \sin(2x)\cos(\pi x))\end{aligned}$$

Finally, we can plug this back into our original integration by parts equation to get that

$$\begin{aligned}\int \sin^2(x)\cos(\pi x)dx &= \frac{1}{\pi}\sin^2(x)\sin(\pi x) - \frac{1}{\pi}\int \sin(2x)\sin(\pi x)dx \\ &= \frac{1}{\pi}\sin^2(x)\sin(\pi x) - \frac{1}{\pi^2-4}(\frac{2}{\pi}\sin(\pi x)\cos(2x) - \sin(2x)\cos(\pi x))\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\pi}\cdot\frac{1}{2}(1-\cos(2x))\sin(\pi x) - \frac{1}{\pi^2-4}\cdot\frac{2}{\pi}\sin(\pi x)\cos(2x) + \frac{1}{\pi^2-4}\sin(2x)\cos(\pi x) \\
&= \frac{1}{2\pi}\sin(\pi x) - (\frac{1}{2\pi} + \frac{1}{\pi^2-4}\cdot\frac{2}{\pi})\sin(\pi x)\cos(2x) + \frac{1}{\pi^2-4}\sin(2x)\cos(\pi x).
\end{aligned}
$$

To make this look more like the answer from the first method, we recall that

$$\frac{1}{\pi^2-4} = \frac{1}{4}(\frac{1}{\pi-2} - \frac{1}{\pi+2})$$

, and that finding a common denominator shows that

$$
\begin{aligned}
\frac{1}{2\pi} + \frac{1}{\pi^2-4}\cdot\frac{2}{\pi} &= \frac{\pi}{2(\pi^2-4)} \\
&= \frac{1}{4}(\frac{1}{\pi-2} + \frac{1}{\pi-2}).
\end{aligned}
$$

Thus we can rewrite

$$-(\frac{1}{2\pi} + \frac{1}{\pi^2-4}\cdot\frac{2}{\pi})\sin(\pi x)\cos(2x) + \frac{1}{\pi^2-4}\sin(2x)\cos(\pi x)$$

as

$$
\begin{aligned}
&\frac{1}{4}[\frac{1}{\pi-2}(\sin(2x)\cos(\pi x) - \cos(2x)\sin(\pi x)) - \frac{1}{\pi+2}(\sin(2x)\cos(\pi x) + \cos(2x)\sin(\pi x))] \\
&\qquad = \frac{1}{4}(\frac{\sin((2-\pi)x)}{\pi-2} - \frac{\sin((\pi+2)x)}{\pi+2}) \\
&\qquad = -\frac{1}{4}(\frac{\sin((\pi-2)x)}{\pi-2} - \frac{\sin((\pi+2)x)}{\pi+2})
\end{aligned}
$$

where the second line comes from the angle addition formulas. Finally, we get that

$$
\begin{aligned}
\int \sin^2(x)\cos(\pi x)dx &= \frac{1}{2\pi}\sin(\pi x) + \frac{1}{4}(\frac{\sin((\pi-2)x)}{\pi-2} - \frac{\sin((\pi+2)x)}{\pi+2}) + C \\
&= -\frac{1}{4}[\frac{\sin((2+\pi)x)}{2+\pi} + \frac{\sin((\pi-2)x)}{\pi-2} - \frac{2}{\pi}\sin(\pi x)] + C
\end{aligned}
$$

as before.

(10 points) 3. Compute the antiderivative

$$\int \frac{6x^2+4x+34}{(x+7)(x^2-2x+12)}dx$$

Solution: Call the integrand $f(x)$. Then $f(x)$ is proper, since its numerator has degree 2 and its denominator has degree 3. Also, $x^2-2x+12$ is irreducible (we can see this by completing the square or using the quadratic formula to check that its roots are not real), so we know its partial fraction decomposition has the form

$$f(x) = \frac{A}{x+7} + \frac{Bx+C}{x^2-2x+12}.$$

Finding a common denominator, this means that

$$6x^2+4x+34 = A(x^2-2x+12) + (Bx+C)(x+7).$$

There are many ways to solve for A, B, and C from here.

<u>Method 1 (Plug in values):</u>
Set $x=-7$. Then

$$6 \cdot 49 - 4 \cdot 7 + 34 = A(7^2 + 2 \cdot 7 + 12) + (C - 7B)(0),$$

so

$$300 = 75A;$$

ie,

$$A = 4.$$

Next, we can plug in $x=0$ to find

$$\begin{aligned} 34 &= 4 \cdot 12 + 7C \\ \Rightarrow -14 &= 7C \\ \Rightarrow -2 &= C. \end{aligned}$$

Lastly, (or by comparing the x^2 coefficients), we could set $x=1$ to find that

$$\begin{aligned} 44 &= 44 + (B-2) \cdot 8 \\ \Rightarrow 0 &= (B-2) \cdot 8 \\ \Rightarrow B &= 2. \end{aligned}$$

<u>Method 2 (Compare coefficients):</u>
Rewrite the first equation as

$$\begin{aligned} 6x^2+4x+34 &= A(x^2-2x+12) + (Bx+C)(x+7) \\ &= (A+B)x^2 + (-2A+7B+C)x + (12A+7C), \end{aligned}$$

which means A, B, C satisfy the linear equations

$$A + B = 6, -2A + 7B + C = 4, 12A + 7C = 34.$$

We can solve these to get

$$A = 4, B = 2, C = -2$$

again.

Now we can integrate:

$$\begin{aligned}\int \frac{6x^2 + 4x + 34}{(x+7)(x^2 - 2x + 12)} dx &= \int \frac{4}{x+7} + \frac{2x-2}{x^2 - 2x + 12} dx \\ &= 4 \log|x+7| + \log|x^2 - 2x + 12| + C,\end{aligned}$$

where for the last equality we used the substitution $u = x^2 - 2x + 12$ to transform

$$\int \frac{2x-2}{x^2 - 2x + 12} dx$$

into

$$\int \frac{1}{u} du.$$

(2 points) 4. (a) Find the general form of the partial fraction decomposition for the rational function

$$\frac{x^3}{(x+11)^4(x^2+29)^2}$$

but do not solve for the actual constants involved.

Solution:

$$\frac{x^3}{(x+11)^4(x^2+29)^2} = \frac{A}{(x+11)^4}+\frac{B}{(x+11)^3}+\frac{C}{(x+11)^2}+\frac{D}{x+11}+\frac{Fx+G}{(x^2+29)^2}+\frac{Hx+I}{x^2+29}$$

(8 points) (b) Compute the integral

$$\int_0^\infty \frac{2}{(x^2+1)^2}dx$$

(Hint: this integral converges; use trig substitution)

Solution: Finding an antiderivative of this was essentially an example in class. We make the substitution

$$x = \tan^2(\theta)$$

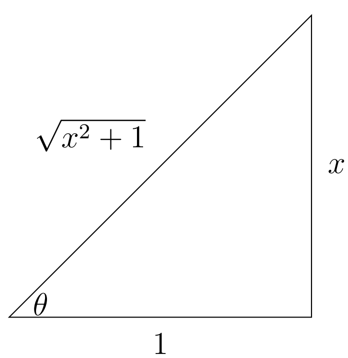
to exploit the identity

$$\tan^2(\theta) + 1 = \sec^2(\theta).$$

As $dx = \sec^2(\theta)d\theta$, the substitution becomes

$$\begin{aligned}
\int \frac{2}{(x^2+1)^2}dx &= 2\int \frac{\sec^2(\theta)}{(\tan^2(\theta)+1)^2}d\theta \\
&= 2\int \frac{\sec^2(\theta)}{(\sec^4(\theta)}d\theta \\
&= 2\int \cos^2(\theta) \\
&= 2\int \frac{1}{2}(1+\cos(2\theta))d\theta \\
&= \theta + \frac{1}{2}\sin(2\theta) + C \\
&= \theta + \cos(\theta)\sin(\theta) + C.
\end{aligned}$$

If we wanted this as a function of x (not strictly necessary in this case because we could just figure out what the bounds are for θ), we would draw a right triangle



which shows that

$$\sin(\theta) = \frac{x}{\sqrt{x^2+1}}, \cos(\theta) = \frac{1}{\sqrt{x^2+1}}.$$

Thus we have that

$$\int \frac{2}{(x^2+1)^2}dx = \arctan(x) + \frac{x}{x^2+1} + C.$$

Now we can calculate the definite integral:

$$\begin{aligned}
\int_0^\infty \frac{2}{(x^2+1)^2}dx &= \lim_{R\to\infty}\int_0^R \frac{2}{(x^2+1)^2}dx \\
&= \lim_{R\to\infty}\left[\arctan(x)+\frac{x}{x^2+1}\right]_0^R \\
&= \lim_{R\to\infty}\left[\arctan(R)+\frac{R}{R^2+1}-\arctan(0)\right] \\
&= \lim_{R\to\infty}\arctan(R)+\frac{R}{R^2+1} \\
&= \frac{\pi}{2}+0 \\
&= \frac{\pi}{2}.
\end{aligned}$$

(6 points) 5. (a) Show that

$$\int_9^\infty \frac{1}{x\sqrt{x^2-81}}dx$$

converges.

Solution: Again, we can use trig substitution to find the antiderivative. If we let $x = 9\sec(\theta)$, then

$$\begin{aligned}\int \frac{1}{x\sqrt{x^2-81}}dx &= \int \frac{9\sec(\theta)\tan(\theta)}{9\sec(\theta)\sqrt{81\tan^2(\theta)}}d\theta \\ &= \frac{1}{9}\frac{\sec(\theta)\tan(\theta)}{\sec(\theta)\tan(\theta)}d\theta \\ &= \frac{1}{9}\operatorname{arcsec}(\theta) + C \\ &= \frac{1}{9}\operatorname{arcsec}(\frac{x}{9}) + C.\end{aligned}$$

It turns out that this improper integral is improper on both sides, because $x^2 - 81 = 0$ when $x = 9$. Thus, we need to take limits on both sides:

$$\begin{aligned}\int_9^\infty \frac{1}{x\sqrt{x^2-81}}dx &= \frac{1}{9}(\lim_{R\to\infty}\operatorname{arcsec}(\frac{R}{9}) - \lim_{S\to 9^+}\operatorname{arcsec}(\frac{S}{9})) \\ &= \frac{1}{9}(\frac{\pi}{2} - 0) \\ &= \frac{\pi}{18}.\end{aligned}$$

Therefore, the integral converges.

(4 points) (b) Does

$$\int_9^\infty \frac{1}{x\sqrt{\log(x)x^2-81}}dx$$

converge? (Hint: $9 > e$)

Solution: Notice that $\log(x) > 1$, because (for example) $x > 9 > e \sim 2.7$ and taking log preserves inequalities. This means that

$$\log(x)x^2 - 81 > x^2 - 81,$$

and the same is true when we take square roots:

$$\sqrt{\log(x)x^2-81} > \sqrt{x^2-81}.$$

When we divide by both sides, the inequality flips, so

$$\frac{1}{\sqrt{\log(x)x^2-81}} < \frac{1}{\sqrt{x^2-81}},$$

meaning

$$\frac{1}{x\sqrt{\log(x)x^2-81}} < \frac{1}{x\sqrt{x^2-81}}.$$

Thus, by comparison to the integral from part a, which we just showed converges,

$$\int_9^\infty \frac{1}{x\sqrt{\log(x)x^2-81}}dx$$

converges as well.

Scratch paper