Partial Fractions

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These notes give further motivation for the forms of the partial expansions that we have seen in class. The starting point is the following:

Theorem. *If $f(x) = \frac{p(x)}{q(x)}$ is a proper rational function (ie, p and q are polynomials and the degree of p is smaller than that of q), and q has distinct linear factors of multiplicity 1*

$$q(x) = (x - a_1)(x - a_2) \cdots (x - a_n)$$

then $f(x)$ has a partial fraction expansion (PFE)

$$f(x) = \frac{A_1}{x - a_1} + \cdots + \frac{A_n}{x - a_n}$$

for some constants A_1 through A_n.

We sketched a proof of this in class by equating the coefficients of $p(x)$—a degree $\leq n-1$ polynomial since f is proper—to those of the polynomial $q(x)f(x)$ when we multiply the right hand side of the PFE in the theorem. We found that the n different coefficients of p give n linear equations in A_1 through A_n, and remembered that in general n linear equations for n unknown variables will have a unique solution. Thus for any p, q of the given form, we can find $A_1, A_2, \cdots A_n$ making the equation in the partial fraction expansion true.

Of course, not every proper rational function will have q in such a convenient form. In some cases, a little more is required: when we have an irreducible quadratic, or when we have repeated factors.

1 Irreducible real quadratics

Suppose we have already performed any necessary division, so that we start with a rational function $f(x) = \frac{p(x)}{q(x)}$, where p and q are both real polynomials and the degree of p is smaller than that of q. We can get a partial fraction expansion for f by factoring $q(x)$ as a product of powers of real irreducible quadratic and real linear factors. For simplicity, suppose that $q(x)$ just has one quadratic and one linear factor:

$$q(x) = (x - a)(x^2 + bx + c)$$

with a, b, c real. If $x^2 + bx + c$ is irreducible over the reals (so it has no real roots), then it has two complex conjugate roots β and $\bar{\beta}$, hence (by the Fundamental Theorem of Algebra, for example) factors as

$$x^2 + bx + c = (x - \beta)(x - \bar{\beta}).$$

We know from class that f's partial expansion is supposed to have the form

$$\frac{p(x)}{q(x)} = \frac{p(x)}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c};$$

here is one way to explain the presence of the $Bx + C$ on top. If we factor q into (complex) linear factors instead, we would have

$$f(x) = \frac{p(x)}{(x-a)(x-\beta)(x-\bar{\beta})}$$

and our usual partial fraction expansion when we have distinct linear factors would say that

$$f(x) = \frac{A}{(x-a)} + \frac{D}{x-\beta} + \frac{E}{(x-\bar{\beta})}.$$

This works just as well over the complex numbers as it does over the reals (but D and E may not be real anymore), but we want to end up with something real if we started with a real rational function.

Claim. $\frac{D}{x-\beta} + \frac{E}{(x-\bar{\beta})}$ is a real rational function.

Proof. This is true because $\frac{D}{x-\beta} + \frac{E}{(x-\bar{\beta})} = f(x) - \frac{A}{x-a}$ is the difference of two real rational functions, so it should be real as well. To see this explicitly, we can use the equation

$$\begin{aligned} p(x) &= A(x-\beta)(x-\bar{\beta}) + D(x-a)(x-\bar{\beta}) + E(x-a)(x-\beta) \\ &= A(x^2+bx+c) + D(x-a)(x-\bar{\beta}) + E(x-a)(x-\beta). \end{aligned}$$

Now we can start plugging in values to find relationships between A, D, and E. We start with $x = a$, which yields

$$p(a) = A(a^2 + ba + c) + D(0) + E(0).$$

As $a^2 + ba + c$ is real (because b and c are) but not 0 (because then a would be a root, but $x^2 + bx + c$ has no real roots), we can divide by it to get

$$A = \frac{p(a)}{(a^2 + ba + c)},$$

so at least A is real, as we expected.

Next, we plug in $x = \beta$, which amounts to

$$p(\beta) = A(0) + D(\beta - a)(\beta - \bar{\beta}) + E(0),$$

which we can solve for D:

$$D = \frac{p(\beta)}{(\beta - a)(\beta - \bar{\beta})}.$$

Plugging in $x = \bar{\beta}$ instead, we arrive analogously at

$$E = \frac{p(\bar{\beta})}{(\bar{\beta} - a)(\bar{\beta} - \beta)}.$$

Why did we do this? These expressions actually show that $E = \bar{D}$, because (as we discussed when we applied the fundamental theorem of algebra to factor real polynomials) $p(x)$ being real implies that $p(\bar{\beta}) = \overline{p(\beta)}$, and all of the other factors in the expressions are conjugates of each other.

Now we can rewrite the partial fraction as

$$f(x) = \frac{A}{(x-a)} + \frac{D}{x-\beta} + \frac{\bar{D}}{(x-\bar{\beta})}$$

and find a common denominator for the two complex terms. To do this, note that

$$\begin{aligned}
\frac{D}{x-\beta} + \frac{\bar{D}}{(x-\bar{\beta})} &= \frac{D(x-\bar{\beta})}{(x-\beta)(x-\bar{\beta})} + \frac{\bar{D}(x-\beta)}{(x-\beta)(x-\bar{\beta})} \\
&= \frac{Dx + \bar{D}x - D\bar{\beta} - \bar{D}\beta}{x^2+bx+c} \\
&= \frac{(D+\bar{D})x - (D\bar{\beta} + \bar{D}\beta)}{x^2+bx+c}.
\end{aligned}$$

Let $B = D + \bar{D}$ and $C = -(D\bar{\beta} + \bar{D}\beta)$. The B is real, because it is a complex number plus its conjugate. But $\bar{D}\beta = \overline{D\bar{\beta}}$, so C is -1 times a sum of conjugates as well, meaning C is real. Therefore, we have the (real) PFE

$$\frac{p(x)}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$$

that we set out to find. $\square$

2 Repeated linear factors

So far, we have only considered the case when all of the factors of q are distinct. What should we do if this does not happen? To begin, consider the proper (so $\deg(p) < \deg(q) = n+1$) rational function

$$f(x) = \frac{p(x)}{(x-a)^n(x-b)}$$

where $a \neq b, n > 1$. Based off of the other cases we know, we could guess that the PFE of f is

$$f(x) = \frac{A}{(x-a)^n} + \frac{B}{(x-b)}.$$

There are many ways to see that this will not work in general. The course supplement explains this using poles: f has a pole of order n at $x = a$, but the PFE should also reflect the poles of lower order at $x = a$. Another way to see this is by comparing coefficients: if we are to solve

$$p(x) = A(x-b) + B(x-a)^n$$

then we have $n+1$ equations (the maximum degree of p is n, so there are at most $n+1$ nonzero coefficients) for only 2 unknowns, which does not usually admit a solution. Even more concretely, if we expand $(x-a)^n$ we would see that the x^n coefficient of the right hand side is just B, so B will be 0 unless $p(x)$ truly has degree n, not anything lower. This rules out many valid choices for p immediately.

What can we do instead? As the book and the supplement suggest, we have the following:

Theorem. *If f is a proper rational function of the form $f(x) = \frac{p(x)}{(x-a)^n g(x)}$ where $(x-a)$ does not divide $g(x)$ (ie, a is not a root of g) then f's PFE looks like*

$$f(x) = \frac{h(x)}{g(x)} + \frac{B_1}{x-a} + \frac{B_2}{(x-a)^2} + \frac{B_n}{(x-a)^n},$$

where h has lower degree than g.

Proof. For this to be true, we must solve (after clearing the denominators)

$$p(x) = h(x)(x-a)^n + B_1 g(x)(x-a)^{n-1} + \cdots + B_n g(x).$$

Here the unknowns we are solving for are the coefficients of h and B_1 through B_n. Since h is supposed to have smaller degree than g, it has at most $\deg(g)$ nonzero coefficients. Thus, the number of unknowns is $n+\deg(g)$. On the other hand, by the properness assumption, $\deg(p)$ is smaller than the degree of $(x-a)^n g(x)$, which has degree $n+\deg(g)$. Since $deg(p)$ is at most $n+\deg(g)-1$, it has at most $n+\deg(g)$ nonzero coefficients, which allows for $n+\deg(g)$ linear equations. Thus, we have $n+\deg(g)$ linear equations for $n+\deg(g)$ unknowns, which we can solve to get the desired PFE. □

Notice that we genuinely needed all of the powers of $(x-a)$ in the denominator so that we could get enough coefficients to solve all of the equations. This makes sense, because $\frac{B_1}{x-a} + \frac{B_2}{(x-a)^2} + \frac{B_n}{(x-a)^n}$ is the most general form of a rational function with poles only at $x = a$ that has a pole of order n.

3 Repeated quadratic factors

The only mystery left is what to do with repeated quadratic factors. This actually follows from the previous case and the reasoning in section 1. If we have a proper rational function $f(x) = \frac{p(x)}{(x^2+bx+c)^n g(x)}$, with x^2+bx+c irreducible over the reals, then we can rewrite f using the factorization $x^2+bx+c = (x-\beta)(x-\bar{\beta})$:

$$f(x) = \frac{p(x)}{(x-\beta)^n (x-\bar{\beta})^n g(x)}.$$

Now we have converted to the case of repeated linear factors, so we know that the PFE is

$$f(x) = \frac{h(x)}{g(x)} + \frac{D_1}{x-\beta} + \frac{D_2}{(x-\beta)^2} + \frac{D_n}{(x-\beta)^n} + \frac{E_1}{x-\bar{\beta}} + \frac{E_2}{(x-\bar{\beta})^2} + \frac{E_n}{(x-\bar{\beta})^n}.$$

The only tricky part, which we will omit, is to check that we can combine terms as in section 1.

Fact. $E_j = \bar{D}_j$ for each j from 1 to n.

Taking this as a given, we can find common denominators and end up with a real PFE in the same way. Altogether, this shows

Theorem. *If f is a proper rational function of the form $f(x) = \frac{p(x)}{(x^2+bx+c)^n g(x)}$ where x^2+bx+c does not divide $g(x)$ then f's PFE looks like*

$$f(x) = \frac{h(x)}{g(x)} + \frac{B_1 x + C_1}{x^2+bx+c} + \frac{B_2 x + C_2}{(x^2+bx+c)^2} + \frac{B_n x + C_n}{(x^2+bx+c)^n}.$$

where h has lower degree than g.

The upshot here is that as soon as we learn how to integrate these types of terms on Monday, we will be able to integrate any rational function.