Math 20B Area between two Polar Curves

Analogous to the case of rectangular coordinates, when finding the area of an angular sector bounded by two *polar* curves, we must subtract the area on the inside from the area on the outside. We know the formula for the area bounded by a polar curve, so the area between two will be

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r_{outer}^2 - r_{inner}^2 d\theta$$

If the two curves are given by $r = f(\theta)$ and $r = g(\theta)$, and $f(\theta) \geq g(\theta) \geq 0$ between the angles α and β, this translates to

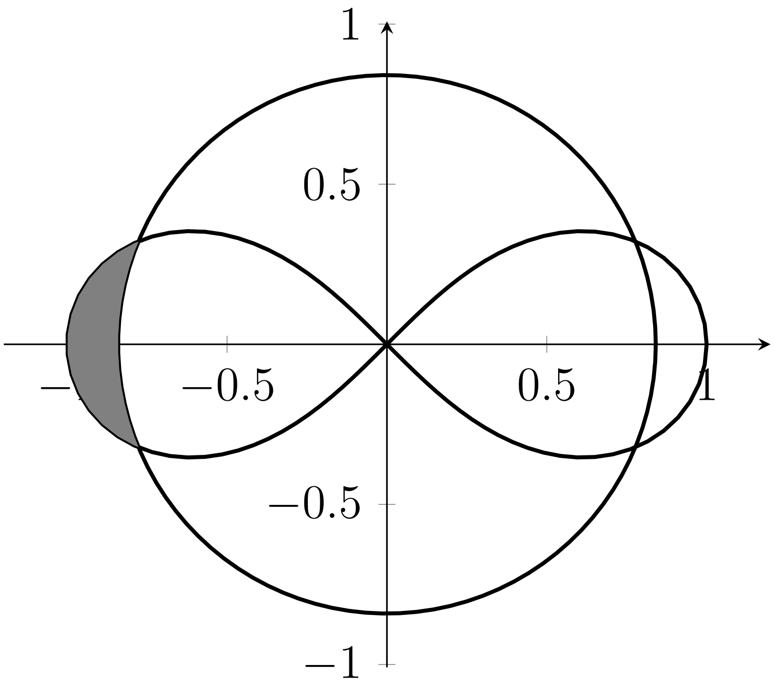
$$A = \frac{1}{2}\int_{\alpha}^{\beta} f(\theta)^2 - g(\theta) d\theta$$

Steps to remember when finding polar area between two curves:

1. Try to draw a picture/sketch a graph of the curves
2. Find the limits of integration (usually by finding the intersection points and identifying the appropriate interval corresponding to the area, often using step 1)
3. Identify the inner and outer curves (often using step 1)
4. Write the integral using our formula
5. Calculate the integral

The last example we found in class was the following:

1. Calculate the shaded area between the circle $r = 2^{-\frac{1}{4}}$ and the lemniscate $r^2 = \cos(2\theta)$



Solution: I provided the graph already, so we can start by finding all the points of intersection between the curves. We can do this by setting the two values of r^2 equal.

$$\begin{aligned}\cos(2\theta) &= r^2 \\ &= (2^{-\frac{1}{4}})^2 \\ &= 2^{-\frac{1}{2}} \\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

This has two solutions up to multiples of 2π:

$$2\theta = \frac{\pi}{4} + 2\pi n, -\frac{\pi}{4} + 2\pi n, n \text{ an integer}$$

and so the possible values for theta are

$$\theta = \frac{\pi}{8} + \pi n, -\frac{\pi}{8} + \pi n, n \text{ an integer.}$$

If we start at $\frac{\pi}{8}$, we can list them in order as

$$\theta = \frac{\pi}{8}, -\frac{\pi}{8} + \pi = \frac{7\pi}{8}, \frac{\pi}{8} + \pi = \frac{9\pi}{8}, -\frac{\pi}{8} = \frac{15\pi}{8}.$$

We don't need to look any further than these 4 because we know angles repeat every 2π.

Looking at the graph, we would like to start in the 2nd quadrant and end in the 3rd quadrant. This means that we should go from $\frac{7\pi}{8}$ to $\frac{9\pi}{8}$ on our list.

From the graph, we can also see that the lemniscate is on the outside between $\frac{7\pi}{8}$ and $\frac{9\pi}{8}$, so the integral should be

$$\frac{1}{2}\int_{\frac{7\pi}{8}}^{\frac{9\pi}{8}} r_{outer}^2 - r_{inner}^2 d\theta = \frac{1}{2}\int_{\frac{7\pi}{8}}^{\frac{9\pi}{8}} \cos(2\theta) - (2^{-\frac{1}{4}})^2 d\theta.$$

Now we're in business: the functions to be integrated are not so bad, and we find that

$$\begin{aligned}
Area &= \frac{1}{2}\int_{\frac{7\pi}{8}}^{\frac{9\pi}{8}} \cos(2\theta) - (2^{-\frac{1}{4}})^2 d\theta \\
&= \frac{1}{2}\int_{\frac{7\pi}{8}}^{\frac{9\pi}{8}} \cos(2\theta) - 2^{-\frac{1}{2}} d\theta \\
&= \frac{1}{2}\left[\frac{1}{2}\sin(2\theta) - 2^{-\frac{1}{2}}\right]_{\frac{7\pi}{8}}^{\frac{9\pi}{8}} \\
&= \frac{1}{2}[\frac{1}{2}(\sin(\frac{9\pi}{4}) - \sin\frac{7\pi}{4}) - 2^{-\frac{1}{2}}(\frac{9\pi}{8} - \frac{7\pi}{8})] \\
&= \frac{1}{2}[\frac{1}{2}(2^{-\frac{1}{2}} - (-2^{-\frac{1}{2}}) - 2^{-\frac{1}{2}}\frac{\pi}{4}] \\
&= \frac{1}{2}[2^{-\frac{1}{2}} - 2^{-\frac{1}{2}}\frac{\pi}{4}] \\
&= \frac{1}{2\sqrt{2}}(1 - \frac{\pi}{4}).
\end{aligned}$$

One thing to notice is that $1 > \frac{\pi}{4}$, so the area is positive, as expected. If we had gotten the inner and outer radii mixed up, one way to know would be if we had ended up with a negative number.