Notes for Section 6.2

As we saw in class, we can use integration to compute volumes as well as areas. The difference is that instead of integrating the height (a 1 dimensional quantity) of a 2D curve to find its area, we integrate the cross sectional area (a 2 dimensional quantity) of a 3D shape to find its volume. The geometric intuition, however, is the same: we break up our object of interest into pieces that become very thin as we take the limit, so that we are essentially integrating something lower dimensional to find something higher dimensional. In both cases, we arrive at the limit of Riemann sums that defines a definite integral. Figure 2 of the first page of this section describes this in a picture, cutting a of a right cylinder into "cylinders of height δy".

Sometimes the shape of the cross sections are given explicitly, and their area can be calculated using geometry. Other times (as in the example of the sphere in class), you need to consider geometry/a picture of the situation to figure out the cross sections too.

Steps to remember when finding volume:

1. Try to draw a picture of the shape, or sketch the graph of the base
2. Figure out which variable you will be integrating
3. Find the limits of integration (use Step 1!)
4. Write the cross sectional area as a function of chosen variable (Use Step 1!)
5. Calculate the integral

Here is a more complicated example, which is also related to the concept of area between curves:

1. (a) What is the volume of the shape with base given by the area enclosed between the curves $y^2 = x^3$, $x + y = 2$, and the x-axis, whose cross sections perpendicular to the y-axis are half circles?

   Solution: Since we are given that the cross sections are perpendicular to the y-axis, we should integrate the area as a function of y. Writing the curves as funcitons of y, we need to see what interval we are integrating, and which of $f(y) = y^{\frac{2}{3}}$ or $g(y) = 2 - y$ is to the right of the other. One bound is given by assumption: it is the x-axis $y = 0$. On the other hand the two curves intersect at solutions to the equation

$$y^{\frac{2}{3}} = 2 - y$$

To solve this, we can cube both sides to get the equation

$$\begin{aligned} y^2 &= (2-y)^3 \\ &= 8 - 12y + 6y^2 - y^3 \end{aligned}$$

which is the same as

$$y^3 - 5y^2 + 12y - 8 = 0.$$

In turn, this cubic factors as

$$(y-1)(y^2 - 4y + 8) = 0$$

where the quadratic has complex roots $y = 2 \pm i$. Thus the only real solution is $y = 1$, so the integral will be from 0 to 1.

This interval only contains one intersection point, so by continuity, one of the curves is to the right of the other through the whole interval. This means we can check which function is larger at any point in $[0, 1]$ except at 1 itself, since we just calculated that they are equal there. A convenient choice is to check at $y = 0$, where

$$f(0) = 0, g(0) = 2 - 0 = 2$$

so we know that $g(y) \geq f(y)$ on $[0, 1]$.

Now we have to calculate the cross sectional area as a function of y. As the cross sections are semicircles, they will have area

$$A(y) = \frac{\pi}{2} r(y)^2$$

where $r(y)$ is the radius of the base (a function of y). The diameter of the semcircle at a given y is the length of the base, which we have just seen is

$$x_{right} - x_{left} = y^{\frac{2}{3}} - (2 - y).$$

This means $r(y) = \frac{1}{2}(y^{\frac{2}{3}} - 2 + y)$, so that

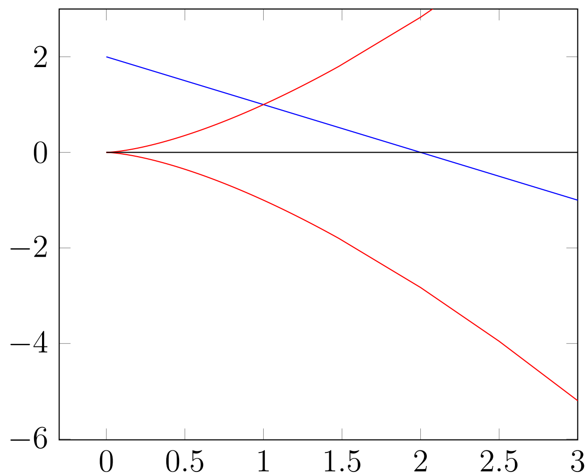
$$A(y) = \frac{\pi}{2}(\frac{1}{2}(y^{\frac{2}{3}} - 2 + y))^2.$$

Therefore, we can calculate the volume as

$$V = \int_0^1 A(y) dy$$

$$= \int_0^1 \frac{\pi}{2}(\frac{1}{2}(y^{\frac{2}{3}} - 2 + y))^2 dy$$
$$= \frac{\pi}{2}\int_0^1 \frac{1}{4}(y^{\frac{4}{3}} + 4 + y^2 - 2y^{\frac{2}{3}} - 2y + y^{\frac{5}{3}})dy$$
$$= \frac{\pi}{8}\int_0^1 y^{\frac{4}{3}} + 4 + y^2 - 2y^{\frac{2}{3}} - 2y + y^{\frac{5}{3}}dy$$
$$= \frac{\pi}{8}\left[\frac{1}{3}y^3 + \frac{3}{8}y^{\frac{8}{3}} + \frac{3}{7}y^{\frac{7}{3}} - \frac{1}{2}y^2 - 2\frac{3}{5}y^{\frac{5}{3}} + 4y\right]_0^1$$
$$= \frac{\pi}{8}(\frac{1}{3} + \frac{3}{8} + \frac{3}{7} - \frac{1}{2} - \frac{6}{5} + 4)$$
$$= \frac{\pi}{8}(\frac{2887}{840})$$

Alternatively, we could have looked at the graph



to help set up the integral. The blue curve is the line $x + y = 2$, and the red the full curve $y^2 = x^3$

(b) What if the cross sections are still semicircles, but perpendicular to the x-axis instead?

Solution: In this case, we must find the area as a function of x instead. We cannot write y as a function of x everywhere for one of our curves because $y^2 = x^3$ two solutions in y for every $x \geq 0$. However, if we restrict only to $y \geq 0$ we can write $y = x^{\frac{3}{2}}$ (that is, always taking the positive square roots of x^3). But we described earlier that the lower y bound for the enclosed area is $y = 0$, meaning the only part of the curve we care about *can* be written as $y = x^{\frac{3}{2}}$. The other curve, of course, is $y = 2 - x$.

In the same way as before, we can see that

$$A(x) = \frac{\pi}{2} r(x)^2$$

where r is half of the diameter (ie, the height) of the base. From the graph, we can see that the height is given by $x^{\frac{3}{2}}$ until the intersection point at $x = 1$, after which it is given by $2 - x$. The bounds with respect to x come from the intersections of the curves with $y = 0$, which are at $x = 0$ and $x = 2$. Thus, the volume is

$$\begin{aligned}
V &= \int_0^2 A(x)dx \\
&= \int_0^1 \frac{\pi}{2}(\frac{1}{2}(x^{\frac{3}{2}}))^2 dx + \int_1^2 \frac{\pi}{2}(\frac{1}{2}(2-x))^2 dx \\
&= \frac{\pi}{8}(\int_0^1 x^3 dx + \int_1^2 (2-x)^2 dx) \\
&= \frac{\pi}{8}(\left[\frac{1}{4}x^4\right]_0^1 + \left[-\frac{1}{3}(2-x)^3\right]_1^2) \\
&= \frac{\pi}{8}(\frac{1}{4} - 0 - 0 + \frac{1}{3}) \\
&= \frac{\pi}{8} \cdot \frac{7}{12} \\
&= \frac{7\pi}{96}
\end{aligned}$$

<u>Remark:</u> The shapes produced in parts a and b are different, because they have different cross sections, so we shouldn't expect their volumes to be the same. As we have seen, they aren't.