Math 200c Spring 2009 Homework 1

Due 4/10/09 in class

All exercise numbers refer to Hungerford. Hand in only exercises marked with a star, though do as many of the other exercises as you can (at the very least, look them over.) All rings are commutative rings with identity in this homework set.

Section III.3: 3*, 4*, 9, 10*

Section III.5: 8*, 9, 10*

Section III.6: 1*, 4, 5, 7(ab)*, 15

Additional problems:

1. Find all solutions to the equation $x^2+y^2 = 2z^2$ in integers $x, y, z \in \mathbb{Z}$ with $\gcd(x, y, z) = 1$. (Work in the Gaussian integers, just as we did in class with $x^2 + y^2 = z^2$; follow the same idea.)

2. Show that $R = \mathbb{Z}[\sqrt{-2}] = \{a + b\sqrt{-2} | a, b \in \mathbb{Z}\}$ is a Euclidean domain, with respect to the norm function $N(a + b\sqrt{-2}) = a^2 + 2b^2$, and hence a UFD. (Hint: Use essentially the same argument as in Exercise III.6, which was assigned in the fall. In order to find $q, r \in R$ such that

$$(a + b\sqrt{-2}) = q(c + d\sqrt{-2}) + r \text{ with } N(r) < N(c + d\sqrt{-2}) = c^2 + 2d^2,$$

examine $(a + b\sqrt{-2})/(c + d\sqrt{-2})$ in $\mathbb{Q}(\sqrt{-2})$ and choose q to be the "closest" element of R to this. Use that the norm function is multiplicative.)

3*. Use a similar method as the one we used in class for the Gaussian integers to describe all irreducible elements of $R = \mathbb{Z}[\sqrt{-2}]$.

4*. Use a similar method as the one we used in class (but working in $R = \mathbb{Z}[\sqrt{-2}]$) to describe all solutions to the equation $x^2+2y^2 = z^2$ in integers $x, y, z \in \mathbb{Z}$ with $\gcd(x, y, z) = 1$.

Math 200c Spring 2009 Homework 2

Due 4/24/09 in class

All exercise numbers refer to Hungerford. Hand in only exercises marked with a star, though do as many of the other exercises as you can (at the very least, look them over.) All rings are commutative rings with identity in this homework set. All subrings are assumed to have the same identity element as the ring in which they are contained.

Section VI.1: 1*, 2(b)*, 5, 6*, 7*, 8

Remarks: I think #5 is too hard to be an exercise, but it is an interesting result and worth thinking about. For #6-8, read over Theorem 1.12 first. For #6(b), note first that given an extension $K \subseteq F$ and a set $S \subseteq F$ which is algebraically independent over K, then every permutation of S induces an automorphism $K(S) \to K(S)$. In fact, one can prove just as easily in #6(b) that there are uncountably many automorphisms of $\mathbb{C}$, if one proves #6(c) first.

Section VIII.1: 2*, 4, 5 (ignore the word "left"), 7*

Additional problems:

1. Let K be a field. The ring $R = K[x_1, x_2, x_3, \dots]$, the polynomial ring in countably many indeterminates, is defined in the obvious way: an element of this ring is a polynomial involving some finite number of the variables. (alternatively, R may be thought of as the union $\bigcup_{n \geq 1} K[x_1, \dots, x_n]$.) Prove that R is not a Noetherian ring.

2*. Let K be a field, and $S = K[x, y]$. Let R the subring $K + xS$ of S. Show that R is not a Noetherian ring.

3*. Show by giving examples that a subring of a Noetherian ring need not be Noetherian, and that a subring of an Artinian ring need not be Artinian.

4*. Let $R \subseteq S$, so R is a subring of S. Consider S as an R-module by multiplication, and suppose that as such S is a finitely generated R-module ($R \subseteq S$ is called an *integral* *extension of rings* in this case.) Prove that if R is a Noetherian ring, then S is a Noetherian ring.

5*. Let R be a ring with ideals I and J such that R/I and R/J are Noetherian rings, and I is a finitely generated ideal. Prove that R/IJ is a Noetherian ring. (Hint: first note that I/IJ is an R/J-module; prove it is a Noetherian module.)

Math 200c Spring 2009 Homework 3

Due 5/8/09 in class

All exercise numbers refer to the handout of sections 15.1-15.3 of Dummit and Foote. Hand in only exercises marked with a star, though do as many of the other exercises as you can (at the very least, look them over.) All rings are commutative rings with identity in this homework set. All subrings are assumed to have the same identity element as the ring in which they are contained.

Section 15.1: 1, 3, 9*, 13, 20*, 21, 22*.

Section 15.2: 2, 5*, 6, 8, 26, 27*.

Additional problems:

1. Let $V \subseteq \mathbb{A}^n$ be a variety (recall this is an *irreducible* closed subset of $\mathbb{A}^n$.) $\mathbb{A}^n$ has the Zariski topology; then V also has a topology given by the subspace topology (the closed sets in V are exactly the sets of the form $X \cap V$ where X is closed in $\mathbb{A}^n$.) Prove that any two nonempty open subsets of V have a nonempty intersection. (We noted this in class only for $V = \mathbb{A}^1$.)

2*. Let $X_1 \supseteq X_2 \supseteq X_3 \supseteq \dots$ be a descending chain of closed subsets of $\mathbb{A}^n$ for some n. Prove that this chain stabilizes, i.e. that there is m such that $X_i = X_m$ for all $i \geq m$.

3*. (a) Let $K = \mathbb{C}$ and let $I = (xy - 1, x^2 + y) \subseteq K[x, y]$. Find $Z(I)$; is $Z(I)$ irreducible? Find $\mathcal{I}(Z(I))$; is it a maximal ideal?

(b). Now let $K = \mathbb{R}$ and do part (a) again.

4*. Let K be any field (not necessarily algebraically closed), and let R be a finitely generated K-algebra. If I is a maximal ideal of R, prove that R/I is a finite-degree field extension of K. (Hint: this result is often considered a variation of the Nullstellensatz—reexamine the proof of that result.) Give an example to show this result may be false if R is a K-algebra which is not a finitely generated K-algebra.

5*. Let R be a commutative domain with field of fractions F. If $0 \neq a \in R$ and $a^{-1} \in F$ is integral over R, then $a^{-1} \in R$ already.

6*. Consider $R = \mathbb{C}[x, y]/(x^2 - y^3)$. ($R$ is the "coordinate ring of a cuspidal cubic curve".)
(1). Show that R is isomorphic to the subring $\mathbb{C}[t^2, t^3]$ of a polynomial ring $\mathbb{C}[t]$.
(2). Show that the integral closure of $\mathbb{C}[t^2, t^3]$ in $\mathbb{C}(t)$ is just $\mathbb{C}[t]$.

(FYI: the fact that R is not integrally closed in its field of fractions (which follows from (1) and (2)) is an algebraic reflection of the geometric fact that the curve $x^2 = y^3$ has a singular point "a cusp" at the origin.)

Math 200c Spring 2009 Homework 4.

Due 6/5/09 in class.

1 Schedule for the remainder of the term

5/22 is our Algebra Qual, 1-4pm in room 6402 AP&M.

- **W 5/13, F 5/15, M 5/18** Class as usual. Finish our discussion of the Wedderburn/Artin theorems, and time permitting begin to discuss representations of groups.
- **W 5/20, F 5/22, M 5/25, W 5/27** NO CLASS on these days. Various people in the class have other quals on 5/20 and 5/27. 5/22 is our Algebra Qual. 5/25 is the Memorial Day holiday.
- **F 5/29, M 6/1, W 6/3, F 6/5** Class as usual. We will give an introduction to the representation theory of groups, and tie together the year with a proof of Burnside's $p^m q^n$-theorem.

2 Final homework

All rings in this homework set have an identity element 1. All exercise numbers refer to the handout from Farb and Dennis's book, "Noncommutative algebra."

Chapter 0 Exercises:
2*, 6, 10, 12, 25*(a,b,c), 42*(b), 49*, 50(a,b).

Chapter 1 Exercises:
#2*, 5*, 6, 8*, 10, 13*(b,c,e)

Note: For #2, do not bother to answer the part of the problem about what happens for rings without 1.

Additional problems:

1*. Suppose that $R = M_n(D)$, where D is a division ring. Let $M = D^n$, the module of column vectors, which is a left R-module where R acts by matrix multiplication. We saw that M is a simple module in class, and so $\mathrm{End}_R(M)$ is a division ring by Schur's Lemma.

Show that for every $d \in D$, the map $\psi_d : M \to M$ given by right multiplication by d, that is, the map

$$\begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{pmatrix} \mapsto \begin{pmatrix} a_1 d \\ a_2 d \\ \dots \\ a_n d \end{pmatrix},$$

is in $\mathrm{End}_R(M)$. Show that every endomorphism of M has the form, and using this, prove that $\mathrm{End}_R(M) \cong D^{op}$ as rings.

2*. We proved in class that the group algebra $\mathbb{C}G$ is semisimple, for any finite group G. So every such group algebra is isomorphic to a finite product of matrix rings over division rings. Find an explicit isomorphism between $\mathbb{C}G$ and a product of matrix rings over division rings, for the symmetric group $G = S_3$. (Hint: try to write $\mathbb{C}G$ as a sum of simple left ideals.)