Consider the ODE:

\[ t^2 y'' + 2ty' - 6y = 0, \quad t > 0. \]

Given that \( y_1(t) = t^2 \) is a solution to the differential equation, use the method of Reduction of Order to find a second solution.

**Solution:** Take a guess for the general solution of the form \( y = v(t)t^2 \). This gives:

\[ y' = v't^2 + 2tv, \quad y'' = t^2v'' + 4tv' + 2v. \]

Plug into the equation:

\[ t^2(t^2v'' + 4tv' + 2v) + 2t(v't^2 + 2tv) - 6vt^2 = 0, \]

we simplify to get:

\[ t^4v'' + 6tv' = 0. \]

Since the \( t > 0 \) we can simplify to get:

\[ v'' + \frac{6}{t}v' = 0. \]

This is a first-order equation in \( v' \). We can see this by letting \( w = v' \) to get:

\[ w' + \frac{6}{t}w = 0. \]

This has integrating factor:

\[ \mu = e^{\int \frac{6}{t}dt} = e^{6\ln t} = t^6. \]

Substituting this and integrating gives us:

\[ \frac{d}{dt}(t^6w) = 0, \quad \Rightarrow \quad t^6w = c_1. \]

Therefore \( w = v' = c_1t^{-6} \). Integrating gives us: \( v = -\frac{c_1}{5}t^{-5} + c_2 \) and absorbing the constants (i.e. with new \( c_1 \)): 

\[ 1 \]
\[ v = c_1 t^{-5} + c_2 . \]

Since the general solution is of form \( y = vt^2 = (c_1 t^{-5} + c_2) t^2 \) we see that the only new linearly independent piece is \( y_2 = t^{-5} \cdot t^2 = t^{-3} \).

**Problem 2**

Consider the ODE:

\[ t^2 y'' - t(t + 2)y' + (t + 2)y = 2t^3 , \quad t > 0 . \]

Given that \( y_1(t) = t \), \( y_2(t) = te^t \) are solutions to this differential equation, use variation of parameters to find the particular solution \( y_{in} \).

**Solution:** To use the formula for variation of parameters we must first write the equation in standard form:

\[ y'' - \left(1 + \frac{2}{t}\right)y' + \left(\frac{1}{t} + \frac{2}{t^2}\right)y = 2t . \]

Therefore we will set \( g(t) = 2t \) in the formula. Let’s compute the Wronskian of \( y_1, y_2 \):

\[ W(y_1, y_2) = te^t + t^2e^t - te^t = t^2e^t . \]

We have:

\[ u_1 = - \int \frac{y_2(t) \cdot g(t)}{W(y_1, y_2)} dt = - \int \frac{te^t \cdot 2t}{t^2e^t} dt = -2 \int dt = -2t , \]

\[ u_2 = \int \frac{y_1(t) \cdot g(t)}{W(y_1, y_2)} dt = \int \frac{t \cdot 2t}{t^2e^t} dt = 2 \int e^{-t} dt = -2e^{-t} . \]

The particular solution is:

\[ y_{in} = u_1 y_1 + u_2 y_2 = (-2t) t + (-2e^{-t}) te^t = -2t^2 - 2t . \]

Note: \(-2t\) is actually a homogeneous solution to this problem. So the only non-trivial part of the particular solution is \(-2t^2\). You don’t have to mention this on your final exam. The answer \( y_{in} = -2t^2 - 2t \) will still get full credit.

**Problem 3**

Find the general solution to the first-order ODE:

\[ (t^2 + t) \frac{dy}{dt} + ty = t^2 - 4t + 5 , \quad t > 0 . \]

**Solution:** Write the equation in standard form:
\[ \frac{dy}{dt} + \frac{1}{t+1} y = \frac{t^2 - 4t + 5}{t(t+1)}. \]

This has integrating factor:
\[ \mu = e^{\int \frac{1}{t+1} \, dt} = e^{\ln(t+1)} = t + 1. \]

Substituting this gives us:
\[ \frac{d}{dt} ((t+1)y) = \frac{t^2 - 4t + 5}{t} = t - 4 + \frac{5}{t}. \]

Integrating yields:
\[ (t+1)y = \frac{t^2}{2} - 4t + 5 \ln t + C, \]

So the solution is:
\[ y = \frac{t^2 - 4t + 5 \ln t + C}{t+1}. \]

**Problem 4**

(a) Determine if the equation is exact (You do not need to solve the equation.)

\[ 2x^2 y + 2xy + \frac{2}{3} y^3 = -(2xy^2 + y^2) \frac{dy}{dx}. \]

(b) The following equation is exact. Find the general solution in implicit form.

\[ (4x^3 y^4 + \cos x) \, dx + (4x^4 y^3 + 6) \, dy = 0. \]

**Solution:** (a) Rearranging the equation:

\[ (2x^2 y + 2xy + \frac{2}{3} y^3) \, dx + (2xy^2 + y^2) \, dy = 0. \]

Therefore:

\[ M = 2x^2 y + 2xy + \frac{2}{3} y^3, \quad N = 2xy^2 + y^2, \]

\[ \frac{\partial M}{\partial y} = 2x^2 + 2x + 2y^2, \quad \frac{\partial N}{\partial x} = 2y^2. \]

Since \( \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \) the equation is not exact.

**Solution:** (b) We have:
\[ M = 4x^3y^4 + \cos x, \quad N = 4x^4y^3 + 6. \]

Integrating in the appropriate variables yields:

\[ \int M \, dx = \int 4x^3y^4 + \cos x \, dx = x^4y^4 + \sin x + g(x), \]

and

\[ \int N \, dy = \int 4x^4y^3 + 6 \, dy = x^4y^4 + 6y + h(y). \]

Taking the common term \( x^4y^4 \) and each of the functions \( g(x) = \sin x \), \( h(y) = 6y \) we get the solution:

\[ \psi(x, y) = x^4y^4 + \sin x + 6y = C. \]

**Problem 5**

Use the Laplace transform to solve the initial value problem:

\[ y'' + 4y = u_\pi(t) - u_{3\pi}(t), \quad y(0) = 4, \quad y'(0) = 3, \]

where \( u_c(t) \) denotes the unit step function with jump at \( t = c \).

**Solution:** Taking Laplace transforms and solving:

\[ s^2 \mathcal{L}[y] - sy(0) - y'(0) + 4 \mathcal{L}[y] = \frac{e^{-\pi s}}{s} - \frac{e^{-3\pi s}}{s}, \]

\[ (s^2 + 4) \mathcal{L}[y] = 4s + 3 + \frac{e^{-\pi s}}{s} - \frac{e^{-3\pi s}}{s}, \]

\[ \mathcal{L}[y] = \frac{4s + 3}{s^2 + 4} + \frac{1}{s(s^2 + 4)}(e^{-\pi s} - e^{-3\pi s}). \]

Next we take \( \mathcal{L}^{-1} \) of both sides. We need to calculate two terms:

\[ I = \mathcal{L}^{-1}\left[\frac{4s + 3}{s^2 + 4}\right], \quad II = \mathcal{L}^{-1}\left[\frac{1}{s(s^2 + 4)}(e^{-\pi s} - e^{-3\pi s})\right] \]

For term \( I \):

\[ \mathcal{L}^{-1}\left(\frac{4s + 3}{s^2 + 4}\right) = 4\mathcal{L}^{-1}\left(\frac{s}{s^2 + 4}\right) + \frac{3}{2} \mathcal{L}^{-1}\left(\frac{2}{s^2 + 4}\right) = 4\cos(2t) + \frac{3}{2}\sin(2t). \]

For term \( II \) we use the partial fraction decomposition:
\[
\frac{1}{s(s^2 + 4)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4}.
\]

Clearing denominators:

\[
1 = A(s^2 + 4) + (Bs + C)s,
\]

\[
1 = (A + B)s^2 + Cs + 4A.
\]

Setting coefficients equal to each other and solving:

\[
A = \frac{1}{4}, B = -\frac{1}{4}, C = 0.
\]

Therefore we have:

\[
\frac{1}{s(s^2 + 4)} = \frac{1}{4s} - \frac{s}{4(s^2 + 4)}.
\]

Multiplying this by \(e^{-\pi s}\) and \(e^{-3\pi s}\) taking \(\mathcal{L}^{-1}\) we have:

\[
II = \mathcal{L}^{-1}[e^{-\pi s}\left(\frac{1}{4s} - \frac{s}{4(s^2 + 4)}\right)] - \mathcal{L}^{-1}[e^{-3\pi s}\left(\frac{1}{4s} - \frac{s}{4(s^2 + 4)}\right)].
\]

We now use Eq. (2) from Wednesday’s lecture:

\[
u_c(t)f(t - c) = \mathcal{L}^{-1}[e^{-cs}F(s)].
\]

This gives us:

\[
\mathcal{L}^{-1}[e^{-\pi s}\left(\frac{1}{4s} - \frac{s}{4(s^2 + 4)}\right)] = \frac{1}{4}u_{\pi}(t)(1 - \cos 2(t - \pi)),
\]

\[
\mathcal{L}^{-1}[e^{-3\pi s}\left(\frac{1}{4s} - \frac{s}{4(s^2 + 4)}\right)] = \frac{1}{4}u_{3\pi}(t)(1 - \cos 2(t - 3\pi)).
\]

Therefore our answer is:

\[
y(t) = 4\cos(2t) + \frac{3}{2}\sin(2t) + \frac{1}{4}u_{\pi}(t)(1 - \cos 2(t - \pi)) - \frac{1}{4}u_{3\pi}(t)(1 - \cos 2(t - 3\pi)).
\]

Note: You do not have to simplify answers on the final exam. However, for completeness I mention that \(\cos 2t\) is 2\(\pi\)-periodic:

\[
\cos 2(t - \pi) = \cos(2t - 2\pi) = \cos 2t,
\]

\[
\cos 2(t - 3\pi) = \cos(2t - 6\pi) = \cos 2t.
\]
Therefore:
\[ y(t) = 4 \cos(2t) + \frac{3}{2} \sin(2t) + \frac{1}{4} u_{\pi}(t)(1 - \cos 2t) - \frac{1}{4} u_{3\pi}(t)(1 - \cos 2t) . \]

**Problem 6**

Let \( \lambda \) be a constant. Consider the ODE:
\[ y'' - 2xy' + \lambda y = 0 . \]

(a) Find the recursion relation for the power series solution centered at \( x_0 = 0 \).

(b) Find the first six non-zero terms of the series solution centered at \( x_0 = 0 \). Write your final answer in terms of the coefficients \( a_0, a_1 \) and \( \lambda \).

**Solution:** (a) We guess a solution in power series of the form \( y = \sum_{n=0}^{\infty} a_n x^n \). Plugging this into the ODE yields:
\[
\begin{align*}
\sum_{n=2}^{\infty} a_n n(n - 1)x^{n-2} - 2x \sum_{n=1}^{\infty} a_n nx^{n-1} + \lambda \sum_{n=0}^{\infty} a_n x^n &= 0 , \\
\sum_{n=0}^{\infty} a_{n+2}(n + 2)(n + 1)x^n - \sum_{n=1}^{\infty} 2a_n nx^n + \sum_{n=0}^{\infty} \lambda a_n x^n &= 0 .
\end{align*}
\]

All the sums start at \( n = 0 \) except for \( \sum_{n=1}^{\infty} 2a_n nx^n \). However the \( n = 0 \) term for this is \( 2a_0(0)x^0 = 0 \). Therefore:
\[
\begin{align*}
\sum_{n=0}^{\infty} a_{n+2}(n + 2)(n + 1)x^n - \sum_{n=0}^{\infty} 2a_n nx^n + \sum_{n=0}^{\infty} \lambda a_n x^n &= 0 , \\
\sum_{n=0}^{\infty} (a_{n+2}(n + 2)(n + 1) - 2a_n n + \lambda a_n) x^n &= 0 .
\end{align*}
\]

Setting the coefficients equal to zero and solving for \( a_{n+2} \):
\[
\begin{align*}
a_{n+2}(n + 2)(n + 1) - (2n - \lambda) a_n &= 0 , & n = 0, 1, 2, 3, 4, ... \\
a_{n+2} &= \frac{a_n (2n - \lambda)}{(n + 2)(n + 1)} , & n = 0, 1, 2, 3, 4, ...
\end{align*}
\]

**Solution:** (b) We plug in integer values of \( n \) and solve in terms of \( a_0, a_1 \), and \( \lambda \).
\[ n = 0 \, , \quad a_2 = a_0 \frac{-\lambda}{2} , \]
\[ n = 1 \, , \quad a_3 = a_1 \frac{2 - \lambda}{3 \cdot 2} , \]
\[ n = 2 \, , \quad a_4 = a_2 \frac{4 - \lambda}{4 \cdot 3} = a_0 \frac{4 - \lambda}{4 \cdot 3 \cdot 2} , \]
\[ n = 3 \, , \quad a_5 = a_3 \frac{(6 - \lambda)}{5 \cdot 4} = a_1 \frac{(6 - \lambda)(2 - \lambda)}{5 \cdot 4 \cdot 3 \cdot 2} . \]

Therefore:

\[ y(x) = a_0 + a_1 x - a_0 \frac{\lambda}{2} x^2 + a_1 \frac{(2 - \lambda)}{3 \cdot 2} x^3 - a_0 \frac{\lambda(4 - \lambda)}{4 \cdot 3 \cdot 2} x^4 + a_1 \frac{(6 - \lambda)(2 - \lambda)}{5 \cdot 4 \cdot 3 \cdot 2} x^5 + ... \]

Note: This is a famous ODE. It is the Hermite equation leading to the Hermite polynomials which are often used in physics and engineering.

**Problem 7**

Find the general solution for the system:

\[ \mathbf{x}'(t) = \begin{pmatrix} 2 & 3/2 \\ -3/2 & -1 \end{pmatrix} \mathbf{x}(t) . \]

**Solution:** Find the characteristic equation:

\[ 0 = \det(A - \lambda I) = \begin{vmatrix} 2 - \lambda & 3/2 \\ -3/2 & -1 - \lambda \end{vmatrix} = (2 - \lambda)(-1 - \lambda) + \frac{9}{4} \]
\[ = -2 - \lambda + \lambda^2 + \frac{9}{4} = \lambda^2 - \lambda + \frac{1}{4} . \]

Therefore:

\[ (\lambda - \frac{1}{2})^2 = 0 . \]

So \( \lambda = \frac{1}{2} \) is a repeated eigenvalue. Solving for the eigenvalue \( \mathbf{v} \) in the equation \( \mathbf{0} = (A - \lambda I)\mathbf{v} \):

\[ \begin{pmatrix} 3/2 & 3/2 \\ -3/2 & -3/2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} , \quad \Rightarrow \quad v_1 = -v_2 , \quad \Rightarrow \quad \mathbf{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} . \]

Therefore:

\[ \mathbf{x}^{(1)}(t) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{\frac{1}{2} t} . \]

Therefore we seek a second solution of the form:
\[
x^{(2)}(t) = vte^{\frac{1}{2}t} + ne^{\frac{1}{2}t},
\]

with \( v = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \) and the vector \( n \) solving: \((A - \lambda I)n = v\). Plugging in:

\[
\begin{pmatrix}
\frac{3}{2} & \frac{3}{2} \\
-\frac{3}{2} & -\frac{3}{2}
\end{pmatrix}
\begin{pmatrix}
n_1 \\
n_2
\end{pmatrix}
= 
\begin{pmatrix}
-1 \\
1
\end{pmatrix}
\]

This gives us:

\[
\frac{3}{2}n_1 + \frac{3}{2}n_2 = -1, \quad -\frac{3}{2}n_1 - \frac{3}{2}n_2 = 1,
\]

So that: \( n_1 = -\frac{2}{3} - n_2 \). Letting \( n_2 = k \) be any real number this gives us \( n_1 = -\frac{2}{3} - n_2 \). So the generalized eigenvector is:

\[
n = \begin{pmatrix}
-\frac{2}{3} - k \\
k
\end{pmatrix}
= \begin{pmatrix}
-\frac{2}{3} \\
0
\end{pmatrix} + k \begin{pmatrix}
-1 \\
1
\end{pmatrix}.
\]

So for any real number \( k \) we have second linearly independent solution:

\[
x^{(2)}(t) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} te^{\frac{1}{2}t} + \left[ \begin{pmatrix} -\frac{2}{3} \\ 0 \end{pmatrix} + k \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right] e^{\frac{1}{2}t}.
\]

Therefore the general solution is: \( y(t) = c_1 x^{(1)}(t) + c_2 x^{(2)}(t) \). Absorbing the piece with \( k \) into \( c_1 x^{(1)}(t) \) yields:

\[
y(t) = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{\frac{1}{2}t} + c_2 \left[ \begin{pmatrix} -1 \\ 1 \end{pmatrix} te^{\frac{1}{2}t} + \begin{pmatrix} -\frac{2}{3} \\ 0 \end{pmatrix} e^{\frac{1}{2}t} \right].
\]

**Problem 8**

Consider the vector-valued functions:

\[
x(t) = \begin{pmatrix}
-e^{\frac{1}{2}t} \\
\frac{1}{2}t e^{\frac{1}{2}t}
\end{pmatrix}, \quad y(t) = \begin{pmatrix}
-te^{\frac{1}{2}t} - \frac{2}{3}e^{\frac{1}{2}t} \\
te^{\frac{1}{2}t}
\end{pmatrix}.
\]

Show that the set \( S = \{x(t), y(t)\} \) is linearly independent for all \( t \).

**Solution:** The easiest way to do this is to calculate the Wronskian:

\[
W(x(t), y(t)) = \begin{vmatrix}
-e^{\frac{1}{2}t} & -te^{\frac{1}{2}t} - \frac{2}{3}e^{\frac{1}{2}t} \\
e^{\frac{1}{2}t} & te^{\frac{1}{2}t}
\end{vmatrix} = -te^{t} + te^{t} + \frac{2}{3}e^{t} = \frac{2}{3}e^{t}.
\]

Clearly for all \( t \) we have \( \frac{2}{3}e^{t} \neq 0 \). Therefore by a Theorem discussed in class the set \( S = \{x(t), y(t)\} \) is linearly independent for all \( t \).