Homework #10
(due Friday, December 4, in class)

Note: Problems 1-3 can be solved using material covered in class through Wednesday, November 25. Problem 4 requires material that will be discussed on Monday, November 30, and Problem 5 requires material that will be discussed on Wednesday, December 2.

1. (Durrett, Exercise 2.4.2, p. 77) Suppose we install a working light bulb at time 0. Suppose the $i$th light bulb works for time $X_i$, then burns out and remains burned out for time $Y_i$ before being replaced. Assume that $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ are i.i.d. sequences with $0 < E[X_1] < \infty$ and $0 < E[Y_1] < \infty$. Let $R_t$ be the amount of time in $[0,t]$ that we have a working light bulb. Show that as $t \to \infty$, we have $R_t/t \to E[X_i]/(E[X_i] + E[Y_i])$ a.s.

2. (similar to Resnick, Problem 1, p. 234). Let $X_1, X_2, \ldots$ be i.i.d. random variables such that $P(X_n = 1) = P(X_n = -1) = 1/2$ for all $n$. Does
\[
\sum_{n=1}^{\infty} \frac{X_n}{n}
\]
converge almost surely? Prove that your answer is correct.

3. (similar to Resnick, Problem 19, p. 238). Suppose $X_1, X_2, \ldots$ are nonnegative independent random variables. For all $n$, define the random variable $Y_n$ by $Y_n(\omega) = \min\{1, X_n(\omega)\}$. Prove that $\sum_{n=1}^{\infty} X_n$ converges almost surely if and only if $\sum_{n=1}^{\infty} E[Y_n] < \infty$.

4. Suppose $X_1, X_2, \ldots$ are independent random variables, and let $S_n = X_1 + \cdots + X_n$. Assume there is a constant $C < \infty$ such that $\text{Var}(X_n) \leq C$ for all $n$. Suppose $r > 1/2$. Show that
\[
\lim_{n \to \infty} \frac{S_n - E[S_n]}{n^r} = 0 \quad \text{a.s.}
\]

5. Let $X_1, X_2, \ldots$ be independent random variables having the exponential distribution with mean 1. That is, their density is given by
\[
f(x) = \begin{cases} 
  e^{-x} & \text{if } x > 0 \\
  0 & \text{if } x \leq 0
\end{cases}
\]
Let $S_n = X_1 + \cdots + X_n$. Prove that if $a > 1$, then $P(S_n \geq na) \leq (ae^{-(a-1)})^n$. 
