Homework #3
(due Wednesday, October 14, in class)

1. Let $X$ and $Y$ be random variables such that for all real numbers $a$ and $b$ with $a < b$, we have $P(a < X < b) = P(a < Y < b)$. Prove that $X$ and $Y$ have the same distribution.

2. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Suppose $Y_1, \ldots, Y_n$ are random variables. Suppose $A_1, \ldots, A_n$ are disjoint events in $\mathcal{F}$ such that $\bigcup_{j=1}^n A_j = \Omega$. Define $X : \Omega \to \mathbb{R}$ by $X(\omega) = Y_j(\omega)$ for $\omega \in A_j$. Show that $X$ is a random variable.

3. Suppose $X$ is a $\mathbb{R}$-valued random variable. Show that for all $\varepsilon > 0$, there exists a bounded random variable $Y$ such that $P(X \neq Y) < \varepsilon$. (We say a random variable $Y$ is bounded if there exists $K < \infty$ such that $|Y(\omega)| \leq K$ for all $\omega \in \Omega$.)

4. Let $X$ be a random variable with distribution function $F$.
   
   (a) Show that $F$ has at most countably many discontinuities.
   
   (b) Show that there are at most countably many $x \in \mathbb{R}$ such that $P(X = x) > 0$.

5. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be lower semicontinuous if for all $x \in \mathbb{R}$ and all $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $f(y) > f(x) - \varepsilon$. Show that if $f : \mathbb{R} \to \mathbb{R}$ is lower semicontinuous, then $f$ is a measurable function.