1. Suppose $X$ is a nonnegative random variable and $P(X > 0) > 0$. Show that $E[X] > 0$.

2. Suppose $X$ is a random variable such that $E[|X|] < \infty$. Prove that for all $\varepsilon > 0$, there exists a simple random variable $Y$ such that $E[|X - Y|] < \varepsilon$.

3. (Durrett, Exercise 1.6.11, p. 35) Suppose $X$ is a random variable such that $E[|X|^k] < \infty$. Suppose $0 < j < k$. Show that $E[|X|^j] \leq (E[|X|^k])^{j/k}$.

4. (Durrett, Exercise 1.6.6, p. 34) Suppose $X$ is a nonnegative random variable such that $0 < E[X^2] < \infty$. Show that

$$P(X > 0) \geq \frac{(E[X])^2}{E[X^2]}.$$ 

(Hint: Apply the Cauchy-Schwarz Inequality to $X1_{\{X > 0\}}$.)

5. Suppose $X$ is a random variable and $a > 0$.

(a) Show that for all $b > 0$, we have $P(X \geq a) \leq E[(X + b)^2]/(a + b)^2$.

(b) Show that if $E[X] = 0$ and $\text{Var}(X) = \sigma^2$, then

$$P(X \geq a) \leq \frac{\sigma^2}{\sigma^2 + a^2}.$$ 

Also show that for each $a > 0$ and $\sigma^2 > 0$, there is an $X$ for which equality holds.