Homework #5
(due Wednesday, October 30, in class)

1. (Durrett, Exercise 1.6.16, p. 33) Let $X$ be a random variable such that $E[|X|] < \infty$. Suppose $A_1, A_2, \ldots$ are disjoint events such that $\bigcup_{n=1}^{\infty} A_n = A$. Show that
\[
\sum_{n=1}^{\infty} E[X \mathbb{1}_{A_n}] = E[X \mathbb{1}_A].
\]

2. Suppose $X_1, X_2, \ldots$ are random variables such that $X_1 \leq X_2 \leq \ldots$ and $E[X_1] > -\infty$. Prove that if $X_n \to X$ a.s., then
\[
\lim_{n \to \infty} E[X_n] = E[X].
\]

3. Suppose $f : \mathbb{R} \to [0, \infty)$ is a measurable function and $\int_{\mathbb{R}} f(x) \, dx = 1$. For all $A \in \mathcal{B}(\mathbb{R})$, define
\[
\mu(A) = \int_A f(x) \, dx,
\]
where $\int_A f(x) \, dx$ means $\int f(x) \mathbb{1}_A(x) \, dx$ and $dx$ denotes integration with respect to Lebesgue measure. Prove that $\mu$ is a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Note: This result implies that $f$ is the density of some random variable.

4. Suppose $f : \mathbb{R} \to [0, \infty)$ is a measurable function such that $\int_{\mathbb{R}} f(x) \, dx = 1$. Suppose $X$ is a random variable with density $f$, meaning that
\[
P(X \in A) = \int_A f(x) \, dx
\]
for all $A \in \mathcal{B}(\mathbb{R})$. Prove that if $g : \mathbb{R} \to \mathbb{R}$ is measurable and $E[g(X)]$ exists, then
\[
E[g(X)] = \int_{\mathbb{R}} g(x) f(x) \, dx.
\]

5. (Resnick, Exercise 30, p. 164). Suppose $X, Y, X_1, X_2, \ldots$, and $Y_1, Y_2, \ldots$ are random variables satisfying the following conditions:

(a) $0 \leq X_n \leq Y_n$ for all $n$.
(b) $X_n \to X$ a.s. and $Y_n \to Y$ a.s.
(c) $\lim_{n \to \infty} E[Y_n] = E[Y] < \infty$.

Prove that $\lim_{n \to \infty} E[X_n] = E[X]$.