Homework #1
(due Wednesday, January 13, in class)

1. Let $U_1, U_2, \ldots$ be i.i.d. random variables, each having the uniform distribution on $[0,1]$. Let $M_n = \min\{U_1, \ldots, U_n\}$. Let $X$ have the exponential distribution with parameter $\lambda = 1$. Prove that $nM_n \Rightarrow X$.

2. (Durrett, Exercise 3.2.4, p. 101) Let $g : \mathbb{R} \to [0, \infty)$ be a continuous function. Suppose $X, X_1, X_2, \ldots$ are random variables such that $X_n \Rightarrow X$. Show that

$$\liminf_{n \to \infty} E[g(X_n)] \geq E[g(X)].$$

3. (Durrett, Exercise 3.2.2, p. 99) Let $X_1, X_2, \ldots$ be independent random variables with distribution function $F$. Let $M_n = \max_{1 \leq m \leq n} X_m$.

(a) Suppose $\alpha > 0$, and $F(x) = 1 - x^{-\alpha}$ for $x \geq 1$. Suppose $Y_1$ has distribution function $F_1$, where $F_1(x) = \exp(-x^{-\alpha})$ for all $x > 0$. Show that

$$n^{-1/\alpha}M_n \Rightarrow Y_1.$$

(b) Suppose $\beta > 0$, and $F(x) = 1 - |x|^\beta$ for $-1 \leq x \leq 0$. Suppose $Y_2$ has distribution function $F_2$, where $F_2(x) = \exp(-|x|^\beta)$ for all $x < 0$. Show that

$$n^{1/\beta}M_n \Rightarrow Y_2.$$

(c) Suppose $F(x) = 1 - e^{-x}$ for all $x \geq 0$. Suppose $Y_3$ has distribution function $F_3$, where $F_3(x) = \exp(-e^{-x})$ for all $x \in \mathbb{R}$. Show that

$$M_n - \log n \Rightarrow Y_3.$$

Note: the distributions of $Y_1$, $Y_2$, and $Y_3$ are called the Fréchet, Weibull, and Gumbel distributions respectively. All three of these distributions are called extreme value distributions. It is possible to show (but you do not have to) that, up to scaling, these are the only distributions that can arise as limits of random variables of the form $(M_n - b_n)/a_n$.

4. (Durrett, Exercise 3.2.11, p. 105) Let $(X_n)_{n=1}^{\infty}$ be a sequence of integer-valued random variables, and let $X$ be an integer-valued random variable. Show that $X_n \Rightarrow X$ if and only if for all integers $m$,

$$\lim_{n \to \infty} P(X_n = m) = P(X = m).$$