Homework #6
(due Wednesday, February 17, in class)

1. Show that two \( \mathbb{R} \)-valued random variables \( X \) and \( Y \) are independent if and only if
   \[
   E[f(X)|Y] = E[f(X)] \quad \text{a.s.}
   \]
   for all bounded measurable functions \( f : \mathbb{R} \to \mathbb{R} \).

2. (similar to Durrett, Exercise 5.1.14, p. 231) Let \( X \) be a \( \mathbb{R} \)-valued random variable on \((\Omega, \mathcal{F}, P)\), and let \( \mathcal{G} \subset \mathcal{F} \) be a \( \sigma \)-field. Suppose \( Q \) is a regular conditional distribution for \( X \) given \( \mathcal{G} \). Suppose \( f : \mathbb{R} \to \mathbb{R} \) is a measurable function such that \( E[|f(X)|] < \infty \). Show that
   \[
   E[f(X)|\mathcal{G}](\omega) = \int_{\mathbb{R}} f(x) Q(\omega, dx) \quad \text{a.s.}
   \]
   (Hint: start with the case in which \( f \) is an indicator function.)

3. Suppose \( X \) is a \( \mathbb{R} \)-valued random variable with \( E[X^2] < \infty \). Suppose \( \mathcal{G} \) is a \( \sigma \)-field and \( a \in \mathbb{R} \). Show that \( E[X1_{\{X \geq a\}}|\mathcal{G}] \leq \sqrt{E[X^2|\mathcal{G}]} P(X \geq a|\mathcal{G}) \) \quad \text{a.s.}
   (Hint: use regular conditional distributions.)

4. (similar to Durrett, Exercise 5.1.13, p. 230) Suppose \( f : \mathbb{R}^2 \to [0, \infty) \) is a measurable function, and \( X \) and \( Y \) are random variables with joint density \( f \). Let \( g(x) = \int_{\mathbb{R}} f(x, y) dy \), and for simplicity assume \( g(x) > 0 \) for all \( x \in \mathbb{R} \). Let \( h(x, y) = f(x, y)/g(x) \). Now for \( \omega \in \Omega \) and \( B \in \mathcal{B}(\mathbb{R}) \), let
   \[
   Q(\omega, B) = \int_{B} h(X(\omega), y) \, dy.
   \]
   (a) Show that \( g \) is a density for \( X \).
   (b) Show that \( Q \) is a regular conditional distribution for \( Y \) given \( \sigma(X) \).