Homework #7
(due Wednesday, February 26, in class)

1. (similar to Durrett, Exercise 4.4.3, p. 207) Suppose $S$ and $T$ are $(\mathcal{F}_n)_{n=0}^{\infty}$-stopping times. Assume that $S \leq T$.
   
   (a) Show that $\mathcal{F}_S \subset \mathcal{F}_T$.
   
   (b) Show that if $A \in \mathcal{F}_S$ and $U = S 1_A + T 1_{A^c}$, then $U$ is a stopping time.

2. (Durrett, Exercise 4.2.1, p. 193) Suppose $(X_n)_{n=0}^{\infty}$ is a martingale with respect to the filtration $(\mathcal{G}_n)_{n=0}^{\infty}$ and let $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$. Show that $\mathcal{F}_n \subset \mathcal{G}_n$ for all $n$ and $(X_n)_{n=0}^{\infty}$ is a martingale with respect to $(\mathcal{F}_n)_{n=0}^{\infty}$.

3. (similar to Durrett, Exercise 4.4.9, p. 208) Suppose $(X_n)_{n=0}^{\infty}$ and $(Y_n)_{n=0}^{\infty}$ are martingales with respect to $(\mathcal{F}_n)_{n=0}^{\infty}$. Assume $E[X_n^2] < \infty$ and $E[Y_n^2] < \infty$ for all $n$.
   
   (a) Show that for all $n$,
   
   $$E[X_n Y_n - X_0 Y_0] = \sum_{m=1}^{n} E[(X_m - X_{m-1})(Y_m - Y_{m-1})].$$

   (b) Show that if $X_0 = 0$, then for all $n$,

   $$E[X_n^2] = \sum_{m=1}^{n} E[(X_m - X_{m-1})^2].$$

4. (Resnick, Exercise 16, p. 431) Suppose $(X_n)_{n=0}^{\infty}$ and $(Y_n)_{n=0}^{\infty}$ are submartingales with respect to the filtration $(\mathcal{F}_n)_{n=0}^{\infty}$.
   
   (a) Let $Z_n = X_n + Y_n$. Show $(Z_n)_{n=0}^{\infty}$ is a submartingale with respect to $(\mathcal{F}_n)_{n=0}^{\infty}$.
   
   (b) Let $W_n = \max\{X_n, Y_n\}$. Show $(W_n)_{n=0}^{\infty}$ is a submartingale with respect to $(\mathcal{F}_n)_{n=0}^{\infty}$. 