Homework #8
(due Wednesday, March 4, in class)

1. Let $X_1, X_2, \ldots$ be i.i.d. random variables taking values in the set \{-1, 0, 1, 2, \ldots\} such that $P(X_i = 0) < 1$. Let $\mu = E[X_1]$, and assume that $\mu \leq 0$. Let $S_0 = 1$, and let $S_n = 1 + X_1 + \cdots + X_n$ for $n \in \mathbb{N}$. Let $T = \inf\{n : S_n = 0\}$. Show that $T < \infty$ a.s.

2. (similar to Durrett, Exercise 5.7.3, p. 272) Let $(S_n)_{n=0}^{\infty}$ be a simple random walk. That is, $S_0 = 0$ and, for $n \geq 1$, we have $S_n = X_1 + \cdots + X_n$, where $X_1, X_2, \ldots$ are i.i.d. with $P(X_i = 1) = P(X_i = -1) = 1/2$.

   (a) Show that $(S_n^2 - n)_{n=0}^{\infty}$ is a martingale.

   (b) Let $T = \inf\{n : S_n \notin (-a, a)\}$, where $a \in \mathbb{N}$. Show that $E[T] = a^2$.

3. Let $(a_n)_{n=0}^{\infty}$ be any infinite sequence of real numbers, and let $\varepsilon > 0$. Prove that there exists a martingale $(X_n)_{n=0}^{\infty}$ such that $P(X_n = a_n$ for all $n \geq 0) \geq 1 - \varepsilon$.

4. (Durrett, Exercise 5.3.1, p. 240) Let $(X_n)_{n=0}^{\infty}$ be a submartingale such that $X_0 = 0$ and

   $$\sup_{n \geq 0} X_n(\omega) < \infty$$

   for all $\omega \in \Omega$. Let $\xi_n = X_n - X_{n-1}$, and suppose $E[\sup_{n \geq 0} \xi_n^+] < \infty$. Show that $(X_n)_{n=0}^{\infty}$ converges a.s.