Homework #2
(due Wednesday, April 13, in class)

1. Let \((B_t)_{t \geq 0}\) be standard Brownian motion. Let \(X_t = B_t - tB_1\) for \(0 \leq t \leq 1\). Note that \(X_0 = 0\) and \(X_1 = 0\). The process \((X_t)_{0 \leq t \leq 1}\) is called a Brownian bridge.

   (a) Show that \((X_t)_{0 \leq t \leq 1}\) is a Gaussian process, and find its mean function and covariance function.

   (b) Show that if \((X_t)_{0 \leq t \leq 1}\) is a Brownian bridge and \(W_t = (t + 1)X_t/(t+1)\) for all \(t \geq 0\), then \((W_t)_{t \geq 0}\) is standard Brownian motion.

2. Let \((B_t)_{t \geq 0}\) be standard Brownian motion. Fix \(\lambda > 0\). For all \(t \in \mathbb{R}\), let \(X_t = e^{-\lambda t}B_{e^{2\lambda t}}\). A stochastic process with continuous paths and the same finite-dimensional distributions as \((X_t)_{t \in \mathbb{R}}\) is called an Ornstein-Uhlenbeck process.

   (a) Show that \(X_t\) has a standard normal distribution for all \(t \in \mathbb{R}\).

   (b) Calculate \(\text{Cov}(X_s, X_t)\) for all \(s, t \in \mathbb{R}\).

   (c) Show that the Ornstein-Uhlenbeck process is stationary, meaning that for all \(s \in \mathbb{R}\), the process \((X_t+s)_{t \in \mathbb{R}}\) is also an Ornstein-Uhlenbeck process.

3. (similar to Durrett, Exercises 8.3.1 and 8.3.3, p. 366) Let \((F_t)_{t \geq 0}\) be a filtration.

   (a) Suppose \(T\) is an \((F_t)_{t \geq 0}\)-stopping time. For \(n \in \mathbb{N}\), let \(T_n = ([2^n T] + 1)/2^n\), which means \(T_n(\omega) = (m+1)2^{-n}\) whenever \(m2^{-n} \leq T(\omega) < (m+1)2^{-n}\). Show that \(T_n\) is a stopping time.

   (b) Show that if \((T_n)_{n=1}^\infty\) is a sequence of \((F_t)_{t \geq 0}\)-stopping times, then \(\sup_n T_n\) is an \((F_t)_{t \geq 0}\)-stopping time.

   (c) Show that if \((T_n)_{n=1}^\infty\) is a sequence of \((F_t)_{t \geq 0}\)-stopping times and \((F_t)_{t \geq 0}\) is right continuous, then \(\inf_n T_n\) is an \((F_t)_{t \geq 0}\)-stopping time.

4. (similar to Durrett, Exercise 8.3.7, p. 368) Suppose \((X_t)_{t \geq 0}\) is a stochastic process with continuous paths which is adapted to a right continuous filtration \((F_t)_{t \geq 0}\). Suppose that \(T\) is an \((F_t)_{t \geq 0}\)-stopping time. Show that \(X_T \mathbb{1}_{\{T < \infty\}}\) is \(F_T\)-measurable.

   (Hint: First show the result when \(T\) is replaced by the stopping times \(T_n\) from part (a) of Problem 3. Then let \(n \to \infty\).)