Homework #3
(due Wednesday, April 20, in class)

1. (Durrett, Exercise 8.2.2, p. 362) Let \((B_t)_{t \geq 0}\) be standard Brownian motion, and let

\[ p_t(0, y) = \frac{1}{\sqrt{2\pi t}} e^{-y^2/2t},\]

so \(y \mapsto p_t(0, y)\) is the density of \(B_t\) if \(B_0 = 0\). Let \(T_0 = \inf\{s > 0 : B_s = 0\}\). Also define \(L = \sup\{t \leq 1 : B_t = 0\}\), which is the last time the Brownian motion visits 0 before time 1. Show that if \(0 < t < 1\), then

\[ P_0(L \leq t) = \int_{-\infty}^{\infty} p_t(0, y) P_y(T_0 > 1 - t) \, dy. \]

2. (Durrett, Exercise 8.4.1, p. 373) Let \((B_t)_{t \geq 0}\) be standard Brownian motion, and define \(T_a = \inf\{t : B_t = a\}\). Show that if \(a \geq 0\) and \(u < v \leq a\), then

\[ P_0(T_a < t, u < B_t < v) = P_0(2a - v < B_t < 2a - u). \]

3. (Durrett, Exercise 8.2.4, p. 363) Let \((B_t)_{t \geq 0}\) be standard Brownian motion. Suppose that \(f : [0, \infty) \to \mathbb{R}\) is a measurable function such that \(f(t) > 0\) whenever \(t > 0\).
   (a) Show that there exists a constant \(c \in [0, \infty]\) such that \(P_0\)-almost surely, we have

   \[ \limsup_{t \downarrow 0} \frac{B_t}{f(t)} = c. \]

   (b) Show that if \(f(t) = \sqrt{t}\), then \(c = \infty\) (which means almost surely Brownian paths are not Hölder continuous of order 1/2).

4. Let \((B_t)_{t \geq 0}\) be standard Brownian motion. Define the upper right derivative at \(t\) by

\[ D^*B_t = \limsup_{h \downarrow 0} \frac{B_{t+h} - B_t}{h}. \]

(a) Show that for each fixed \(t > 0\), almost surely \(D^*B_t = \infty\).

(b) Show that almost surely there exists a \(t > 0\) such that \(D^*B_t \leq 0\).