Homework #4
(due Wednesday, April 27, in class)

1. Suppose \((X_t)_{t \geq 0}\) is a right-continuous martingale with respect to the filtration \((\mathcal{F}_t)_{t \geq 0}\). Suppose \(1 < p < \infty\). Show that for all \(t > 0\), we have
\[
E\left[ \sup_{0 \leq s \leq t} |X_s|^p \right] \leq \left( \frac{p}{p-1} \right)^p E[|X_t|^p].
\]

2. Let \((X_t)_{t \geq 0}\) be a right-continuous martingale with respect to the filtration \((\mathcal{F}_t)_{t \geq 0}\). Fix \(a < b\). Define the number of upcrossings by
\[
U = \sup\{m : \text{there exist } t_1 < \cdots < t_{2m} \text{ such that } X_{t_{2j-1}} < a \text{ and } X_{t_{2j}} > b \text{ for } j = 1, \ldots, m\}.
\]
Show that \((b-a)E[U] \leq |a| + \sup_{t \geq 0} E[X_t^+].\)
(Hint: show that an approximating discrete-time martingale has at least as many upcrossings as the continuous-time martingale.)

3. (similar to Durrett, Exercise 8.5.2, p. 378) Let \((B_t)_{t \geq 0}\) be standard Brownian motion. Let \(\mu > 0\), and let \(X_t = B_t + \mu t\) for all \(t \geq 0\), so \((X_t)_{t \geq 0}\) is Brownian motion with drift \(\mu\). Let \(T_a = \inf\{t \geq 0 : X_t = a\}\).

(a) Show that if \(a < 0 < b\), then \(P(T_a < T_b) = \frac{1 - e^{-2\mu b}}{e^{-2\mu a} - e^{-2\mu b}}.\)
(Hint: Consider the exponential martingale.)

(b) Deduce that if \(M = \inf_{t \geq 0} X_t\) and \(x > 0\), then \(P(-M \geq x) = e^{-2\mu x}.\) That is, \(-M\) has an exponential distribution with rate parameter \(2\mu\).

4. Let \((B_t)_{t \geq 0}\) be standard Brownian motion. Let \(S_t = \sup_{0 \leq s \leq t} B_s\). Show that for all \(a > 0\) and \(t > 0\), we have
\[
P(S_t \geq at) \leq e^{-a^2 t/2}.
\]
(Hint: Apply Doob’s Maximal Inequality to the exponential martingale.)