1. Recall that a random variable $Y$ is said to have a stable distribution if for all positive integers $n$, there exist real numbers $a_n$ and $b_n$ such that if $Y_1, \ldots, Y_n$ are i.i.d. and have the same distribution as $Y$, then $(Y_1 + \cdots + Y_n - b_n)/a_n$ has the same distribution as $Y$. Prove that if $Y$ has a stable distribution, then there exists a Lévy process $(X_t)_{t \geq 0}$ such that $X_1$ has the same distribution as $Y$.

2. Let $(N_t)_{t \geq 0}$ be a homogeneous Poisson process of rate $\lambda$, meaning that $(N_t)_{t \geq 0}$ is a Lévy process and the distribution of $N_t$ is Poisson with mean $\lambda t$ for all $t \geq 0$. Find the values of $b$, $\sigma^2$, and $\nu$ such that for all $s \in \mathbb{R}$ and $t \geq 0$, we have $E[e^{isN_t}] = e^{t\Psi(s)}$, where

$$\Psi(s) = ibs - \frac{\sigma^2 s^2}{2} + \int_{-\infty}^{\infty} (e^{isx} - 1 - isx1_{[-1,1]}(x)) \nu(dx).$$

3. Suppose $\nu$ is a finite measure on $\mathbb{R}$ such that $\int_{-\infty}^{\infty} |x| \nu(dx) < \infty$, and suppose $(X_t)_{t \geq 0}$ is a Lévy process such that $E[e^{isX_t}] = e^{t\Psi(s)}$, where

$$\Psi(s) = \int_{-\infty}^{\infty} (e^{isx} - 1) \nu(dx).$$

(a) Let $T = \inf\{t : X_t \neq 0\}$, and calculate $P(T > t)$ in terms of $t$ and $\nu$.

(b) Find an expression for $E[X_t]$ in terms of $t$ and $\nu$.

(Hint: Construct $(X_t)_{t \geq 0}$ from a Poisson point process. Because $\nu$ is a finite measure, the construction of Lévy processes that we did in class can be simplified in this case.)

4. Let $(X_t)_{t \geq 0}$ be a subordinator such that $E[e^{-\lambda X_t}] = e^{-t\Phi(\lambda)}$ for all $\lambda \geq 0$ and $t \geq 0$, where

$$\Phi(\lambda) = d\lambda + \int_{0}^{\infty} (1 - e^{-\lambda x}) \nu(dx).$$

Here $d \geq 0$ and $\int_{0}^{\infty} (1 \wedge x) \nu(dx) < \infty$. Show that almost surely

$$\lim_{t \to \infty} \frac{X_t}{t} = d + \int_{0}^{\infty} x \nu(dx).$$