Homework #10
(due Thursday, June 14, at 6:00 PM)

Please drop off your completed homework in the instructor’s office (6157 Applied Physics and Mathematics) before the deadline. The instructor will be there Thursday from 3:00-6:00 PM, as well as most of the day on Monday, Tuesday, and Wednesday.

1. Let \((X_t)_{t \geq 0}\) and \((Y_t)_{t \geq 0}\) be independent standard Brownian motions. Let \(Z_t = X_t - Y_t\) for all \(t \geq 0\).

   (a) Show that \((Z_t)_{t \geq 0}\) is Brownian motion, and find the variance parameter.
   (b) Show that with probability one, \(X_t = Y_t\) for infinitely many \(t\).

2. Let \((B_t)_{t \geq 0}\) be standard Brownian motion, and let \(\tau_x = \min\{t : B_t = x\}\). Calculate \(P(\tau_1 < \tau_{-2} < \tau_3)\).

3. Let \((B_t)_{t \geq 0}\) be standard Brownian motion, and let \(\tau_x = \min\{t : B_t = x\}\). Use the exponential martingale to show that if \(x > 0\), then \(E[e^{-\lambda \tau_x}] = e^{-x \sqrt{2\lambda}}\) for all \(\lambda > 0\).

4. Suppose \((B_t)_{t \geq 0}\) is a standard Brownian motion.

   (a) Find \(P(B_t \leq 1\) for all \(t \leq 4\)).
   (b) Find \(P(B_t < 0\) for all \(t > 20\)).

5. Suppose \((B_t)_{t \geq 0}\) is a standard Brownian motion. Fix \(t > 0\).

   (a) Find \(P(B_s > 0\) for all \(s \in [t, 2t]\)).
   (b) Find \(P(B_{2t} < 0|B_t > 0)\). (Hint: consider the Reflection Principle)