Homework #3
(due Wednesday, April 25, at 4:00 PM)

1. Let \((X_n)_{n=0}^\infty\) be a Markov chain with state space \(S = \{1, 2, 3\}\) and transition matrix

\[
P = \begin{bmatrix}
0 & .5 & .5 \\
1 & 0 & 0 \\
.3 & 0 & .7
\end{bmatrix}.
\]

(a) Find a stationary distribution for this Markov chain.

(b) Is this Markov chain irreducible? Explain your answer.

(c) Is this Markov chain aperiodic? Explain your answer.

2. Suppose \((X_n)_{n=0}^\infty\) is a sequence of independent random variables, each having the Poisson distribution with parameter \(\lambda > 0\), which means \(P(X_n = k) = e^{-\lambda}\lambda^k/k!\) for all nonnegative integers \(k\).

(a) Show that \((X_n)_{n=0}^\infty\) is a Markov chain. Give the state space and transition probabilities.

(b) Find a stationary distribution for \((X_n)_{n=0}^\infty\).

(c) Is the state 3 positive recurrent, null recurrent, or transient? Explain your answer.

3. Consider a Markov chain with state space \(S = \{0, 1, 2, \ldots\}\) whose transition probabilities are given by \(p(0,0) = 0, p(0,k) = 2^{-k}\) for all positive integers \(k\), and \(p(k,0) = 1\) for all positive integers \(k\).

(a) Is this Markov chain irreducible?

(b) For each state in \(S\), indicate whether the state is positive recurrent, null recurrent, or transient. Explain your answer.

(c) Find a stationary distribution for this Markov chain.

4. Suppose \((X_n)_{n=0}^\infty\) is a Markov chain with state space \(S\). Suppose that \(\pi_1\) and \(\pi_2\) are both stationary distributions for this Markov chain. Suppose \(0 < \alpha < 1\). For all \(i \in S\), let \(\pi(i) = \alpha\pi_1(i) + (1 - \alpha)\pi_2(i)\). Show that \(\pi\) is also a stationary distribution.

5. Suppose \((X_n)_{n=0}^\infty\) is a Markov chain with state space \(S\) and transition probabilities \(p(i,j)\). Suppose \(\pi\) is a stationary measure. Prove by induction that for all \(j \in S\) and all positive integers \(n\), we have

\[
\pi(j) = \sum_{i \in S} \pi(i)p_n(i,j).
\]