1. Consider a Galton-Watson branching process started with one individual in generation zero. Assume that each individual independently has two offspring in the next generation with probability $p$ and zero offspring in the next generation with probability $1 - p$, where $0 \leq p \leq 1$.

   (a) For what values of $p$ is this process subcritical? For what values of $p$ is the process critical? For what values of $p$ is the process supercritical?
   
   (b) Find the probability that the process eventually goes extinct, as a function of $p$.

2. Let $X_t$ be the number of customers who have arrived in store 1 by time $t$, and let $Y_t$ be the number of customers who have arrived in store 2 by time $t$. Assume that $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ are independent Poisson processes with rates $\lambda_1$ and $\lambda_2$ respectively.

   (a) Find the probability that the first customer who arrives at one of the two stores goes to store 1.
   
   (b) Find the probability that a total of 3 people arrive at the two stores before time $t$.
   
   (c) Compute $P(X_1 = 2, X_2 = 4, X_3 = 4)$.
   
   (d) If you know that 5 people arrive in store 1 before time 4, what is the probability that 3 people arrived in store 1 before time 2?

3. Suppose $(X_t)_{t \geq 0}$ is a continuous-time Markov chain with state space $S = \{1, 2, 3, 4\}$ and transition rates $q(i, j) = 1$ if $j > i$ and $q(i, j) = 0$ if $j \leq i$. Suppose $X_0 = 1$.

   (a) Calculate $P(X_t = 1)$.
   
   (b) What is the probability that the Markov chain never visits the state 2?
   
   (c) What is $\lim_{t \to 0} t^{-1} P(X_t = 2)$?

4. Let $(X_t)_{t \geq 0}$ be a continuous-time Markov chain with state space $S = \{0, 1, \ldots, N\}$ and transition rates given by $q(i, i + 1) = q(i, i - 1) = i(N - i)/N$ for $i = 1, \ldots, N - 1$ and $q(i, j) = 0$ if either $i \in \{0, N\}$ or $j \notin \{i - 1, i + 1\}$. Assume that $X_0 = 1$. (This process is called the Moran model and is used in population genetics to model the number of type 1 individuals in a population with $N$ individuals, some of type 1 and some of type 2.)

   (a) What is the probability that the Markov chain $(X_t)_{t \geq 0}$ reaches $N$ before 0?
   
   (b) Suppose $N = 4$. On average, how much time elapses before the chain reaches 0 or $N$?

5. Let $(X_t)_{t \geq 0}$ be a continuous-time Markov chain with state space $S$. For $t \geq 0$ and $i, j \in S$, let $p_t(i, j) = P(X_t = j | X_0 = i)$. Show that if $q(i, j) > 0$, then $p_t(i, j) > 0$ for all $t > 0$. 

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**Homework #5**

(due Wednesday, May 9, at 4:00 PM)