Problem Set #1

(due Wednesday, October 6, in class)

1. Let $(B^1_t)_{t\geq 0}, \ldots, (B^m_t)_{t\geq 0}$ be independent one-dimensional Brownian motions, with $B^i_0 = 0$ for $i = 1, \ldots, m$. Let $a_1, \ldots, a_m \in \mathbb{R}$. For $t \geq 0$, let $W_t = a_1 B^1_t + \cdots + a_m B^m_t$. Show that if $a_1^2 + \cdots + a_m^2 = 1$, then $(W_t)_{t\geq 0}$ is one-dimensional Brownian motion.

2. Let $(B_t)_{t\geq 0}$ be one-dimensional Brownian motion with $B_0 = 0$. Assume that $(B_t)_{t\geq 0}$ is adapted to the standard filtration $(\mathcal{F}_t)_{t\geq 0}$. Let $a > 0$, and let $T = \inf\{t : B_t > a\}$. Prove that T is a stopping time with respect to $(\mathcal{F}_t)_{t\geq 0}$.

3. (Oksendal, Exercise 3.5, p. 38) Let $(B_t)_{t\geq 0}$ be one-dimensional Brownian motion with $B_0 = 0$. Let $(\mathcal{F}_t)_{t\geq 0}$ be the associated standard filtration. Show that $(B_t^2 - t)_{t\geq 0}$ is a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$.

4. (similar to Oksendal, Exercise 3.3, p. 38)

a) Suppose $(M_t)_{t\geq 0}$ is a martingale with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$. For all $t \geq 0$, let $\mathcal{G}_t = \sigma((M_s)_{0\leq s\leq t})$. Show that $(M_t)_{t\geq 0}$ is a martingale with respect to the filtration $(\mathcal{G}_t)_{t\geq 0}$.

b) Give an example of a stochastic process $(X_t)_{t\geq 0}$ such that $E[X_0] = E[X_t] = 0$ for all $t \geq 0$ but $(X_t)_{t\geq 0}$ is not a martingale with respect to any filtration.