Problem Set #4

(due Wednesday, October 27, in class)

1. (similar to Chung-Williams, Exercise 1, pp. 90-91) Suppose that M is a continuous martingale that is L^2-bounded, meaning there exists a positive number C such that $E[M_t^2] \leq C$ for all $t \geq 0$. Let $\langle M \rangle_\infty = \lim_{t\to\infty} \langle M \rangle_t$. Show that $E[\langle M \rangle_\infty] < \infty$ and that $\lim_{t\to\infty} \int_0^t M \, dM$ exists a.s. and has finite mean.

2. (similar to Durrett, Exercise 3.5, p. 51) Suppose M is a continuous local martingale and T is a stopping time. Let M^T denote the local martingale defined by $M_t^T = M_{T\wedge t}$. Show that almost surely $\langle M^T \rangle_t = \langle M \rangle_{T \wedge t}$ for all $t \geq 0$.

3. Suppose M is a continuous local martingale, and suppose $0 < s < t$. Show that

$$P(\langle M \rangle_s = \langle M \rangle_t \text{ and } M_s \neq M_t) = 0.$$

Hint: Let $T = \inf\{u > s : \langle M \rangle_u > \langle M \rangle_s\} \wedge t$, and show that $M_s = M_T$ a.s.

4. (Durrett, Exercise 6.7, p. 67) Fix $t > 0$, and suppose $f : [0, t] \to \mathbb{R}$ is a continuous function. Show that

$$\int_0^t f(s) \, dB_s$$

has a normal distribution with mean zero and variance $\int_0^t f(s)^2 \, ds$.

Note: The integral should be interpreted as $\int_0^t X \, dB$, where $X_s(\omega) = f(s)$ for all $\omega \in \Omega$.