Homework #5  
(due Wednesday, November 12, in class)

1. (Chung-Williams, Exercise 7, p. 114) Let $X$ and $Y$ be continuous semimartingales. Fix $t > 0$, and let $(\Delta_n)_{n=1}^{\infty}$ be a sequence of partitions of $[0,t]$ such that $\lim_{n \to \infty} |\Delta_n| = 0$. Let $(t^n_j)_{j=0}^{k_n}$ be the points of $\Delta_n$. Show that

$$\sum_{j=1}^{k_n} (X_{t^n_j} - X_{t^n_{j-1}})(Y_{t^n_j} - Y_{t^n_{j-1}})$$

converges in probability to $\langle X, Y \rangle_t$ as $n \to \infty$.

2. (Durrett, Exercise 6.7, p. 67) Fix $t > 0$, and suppose $f : [0, t] \to \mathbb{R}$ is a continuous function. Show that

$$\int_0^t f(s) dB_s$$

has a normal distribution with mean zero and variance $\int_0^t (f(s))^2 ds$.

Note: The integral should be interpreted as $\int_0^t X_s dB_s$, where $X_s(\omega) = f(s)$ for all $\omega \in \Omega$.

3. Let $(B_t)_{t \geq 0}$ be one-dimensional Brownian motion with $B_0 = 0$.
   a) Show that

$$B^3_t = 3 \left( \int_0^t B_s ds + \int_0^t B_s^2 dB_s \right).$$

b) Let $\alpha \in \mathbb{R}$, and let $Y_t = e^{\alpha B_t}$ for all $t \geq 0$. Show that, for all $t \geq 0$, we have

$$Y_t = 1 + \frac{\alpha^2}{2} \int_0^t Y_s ds + \alpha \int_0^t Y_s dB_s.$$

4. Let $(B_t)_{t \geq 0}$ be one-dimensional Brownian motion with $B_0 = 0$.
   a) Show that if $(X_t)_{t \geq 0}$ is a process in $L^2_{\text{loc}}(B)$ and

$$E \left[ \int_0^t X_s^2 ds \right] < \infty$$

for each fixed $t > 0$, then the process $X \cdot B$ is a martingale.

b) Let $\lambda \in \mathbb{R}$, and let $X_t = B_t - \lambda t/2$ for all $t$. Apply Itô’s Formula to the semimartingale $(X_t)_{t \geq 0}$ to show that the process $(e^{\lambda B_t - \lambda^2 t/2})_{t \geq 0}$ is a martingale.

Hint: For part a), apply problem 2 of Homework #4 to the process $X$ truncated at time $t$. 