Problem Set #6

(due Wednesday, November 10, in class)

1. (Oksendal, Exercise 4.11c, p. 59) Suppose $(B_t)_{t \geq 0}$ is one-dimensional Brownian motion with $B_0 = 0$. Let $X_t = (B_t + t)e^{-B_t - t/2}$. Show that $(X_t)_{t \geq 0}$ is a martingale.

2. A process $(X_t)_{t \geq 0}$ is called an *Itô process* if it can be expressed in the form

$$X_t = X_0 + \int_0^t R_s \, ds + \int_0^t Z_s \, dB_s,$$

where $B = (B_t)_{t \geq 0}$ is one-dimensional Brownian motion, $Z \in \Lambda(\mathcal{P}, B)$, and $R = (R_t)_{t \geq 0}$ is a predictable process such that $\int_0^t |R_s| \, ds < \infty$ a.s. for all $t \geq 0$. Show that if $f : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable, then

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) Z_s \, dB_s + \int_0^t f'(X_s) R_s \, ds + \frac{1}{2} \int_0^t f''(X_s) Z_s^2 \, ds.$$

3. For continuous semimartingales X and Y, define the Stratonovich integral of X with respect to Y by

$$\int_0^t X \circ dY = \int_0^t X \, dY + \frac{1}{2} \langle X, Y \rangle_t.$$

Now suppose $X = (X^1, \dots, X^d)$ is a vector of semimartingales, and suppose $f : \mathbb{R}^d \to \mathbb{R}$ has continuous first, second, and third partial derivatives. Show that

$$f(X_t) = f(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(X_s) \circ dX_s^i.$$

4. Suppose X and Y are continuous local martingales. Suppose $(\pi_n)_{n=1}^\infty = (\{t_0^n, \dots, t_{k_n}^n\})_{n=1}^\infty$ is a sequence of partitions of $[0, t]$ such that $\lim_{n \to \infty} \delta\pi_n = 0$. Show that as $n \to \infty$,

$$\sum_{j=0}^{k_n - 1} \left(\frac{1}{2} X_{t_j^n} + \frac{1}{2} X_{t_{j+1}^n} \right) (Y_{t_{j+1}^n} - Y_{t_j^n}) \to_p \int_0^t X \circ dY.$$

Note: In Problem 4, if $\frac{1}{2} X_{t_j^n} + \frac{1}{2} X_{t_{j+1}^n}$ were replaced by $X_{t_j^n}$, then we have shown that the sums converge in probability to the Itô integral. The Stratonovich integral is obtained by averaging the values of X at the left and right endpoints, rather than using the left endpoint. The Stratonovich integral has the advantage that, as can be seen from Problem 3, formulas from ordinary calculus are valid. As a result, the Stratonovich integral is useful for studying stochastic processes on manifolds. However, the Itô integral has the advantage of being a local martingale.