Problem Set #7

(due Wednesday, November 17, in class)

1. Suppose M is a continuous local martingale. Suppose there is a predictable process $(H_t)_{t \geq 0}$ such that for all $t \geq 0$, we have $P(H_t > 0) = 1$ and $\langle M \rangle_t = \int_0^t H_s \, ds$ a.s. Show that there is a Brownian motion B such that

$$M_t = M_0 + \int_0^t H_s^{1/2} \, dB_s.$$

2. (similar to Chung-Williams, Exercise 6.2a, p. 137) Let $M = (M^1, \ldots, M^d)$ be a vector of d continuous local martingales. Assume that for all $t \geq 0$, we have $\langle M^i \rangle_t = t$ for $i = 1, \ldots, d$ and $\langle M^i, M^j \rangle_t = 0$ for $i \neq j$. Show that M is a d-dimensional Brownian motion.

Hint: it suffices to show that if $0 \leq s < t$ and $a_1, \ldots, a_d \in \mathbb{R}$, then

$$E[e^{i(a_1(M_t^1 - M_s^1) + \cdots + a_d(M_t^d - M_s^d))} | \mathcal{F}_s] = e^{-(a_1^2 + \cdots + a_d^2)(t-s)/2}.$$

3. Suppose M is a continuous local martingale such that $M_0 = 0$. Suppose there is a deterministic function $f : [0, \infty) \to [0, \infty)$ such that $\lim_{t \to \infty} f(t) = \infty$ and $\langle M \rangle_t = f(t)$ a.s. for all $t \geq 0$.

a) Show that M is a Gaussian process, meaning that if $0 \leq t_0 < t_1 < \cdots < t_k$, then the distribution of $(M_{t_0}, M_{t_1}, \ldots, M_{t_k})$ is multivariate normal.

b) Show that M has independent increments, meaning that if $0 \leq t_0 < t_1 < \cdots < t_k$, then the random variables $(M_{t_1} - M_{t_0}, M_{t_2} - M_{t_1}, \ldots, M_{t_k} - M_{t_{k-1}})$ are independent.