Problem Set #8

(due Wednesday, November 24, in class)

1. (similar to Chung-Williams, Exercise 7.3, p. 156) Let $(B_t)_{t\geq 0}$ be one-dimensional Brownian motion with $B_0 = 0$, and let $L(t, 0)$ denote the local time at zero that the Brownian motion has accumulated up to time t. Let $T = \inf\{t : |B_t| = y\}$. Show that $L(T, 0)$ has an exponential distribution with mean y.

2. Suppose $(Z_t)_{t\geq 0}$ is a stochastic process having continuous paths such that $Z_t > 0$ for all $t \geq 0$ almost surely. Show that $(Z_t)_{t\geq 0}$ is a local martingale if and only if there exists a continuous local martingale $(M_t)_{t\geq 0}$ such that $Z_t = \exp(M_t - \frac{1}{2}\langle M\rangle_t)$ for all $t \geq 0$ almost surely.

3. a) Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $(\mathcal{F}_t)_{t\geq 0}$ be a standard filtration. Let Q be a probability measure that is equivalent to P, and let $\rho = dQ/dP$. Let $\rho_t = E[\rho|\mathcal{F}_t]$ for all $t \geq 0$. Show that if T is a stopping time with respect to $(\mathcal{F}_t)_{t\geq 0}$ and $g : [0, \infty) \to \mathbb{R}$ is a bounded measurable function, then

$$\int_\Omega g(T)\, dQ = E[g(T)\rho_T].$$

b) Let $(B_t)_{t\geq 0}$ be one-dimensional Brownian motion with $B_0 = 0$, and let $(X_t)_{t\geq 0} = (B_t + \mu t)_{t\geq 0}$ be Brownian motion with drift. Let $a > 0$, and let $T = \inf\{t : X_t = a\}$. Calculate the probability density function for T.

Hint: Use part a) to calculate $P(T \leq t)$, and combine this result with the well-known fact about Brownian motion that if $U = \inf\{t : B_t = a\}$, then U has density

$$f(t) = \frac{a}{\sqrt{2\pi t^3}} e^{-a^2/2t}, \quad t \geq 0.$$