Problem Set #9

(due Friday, December 3, in class)

1. Let $B = (B^1, B^2)$ be two-dimensional Brownian motion with $B_0 = (0,0)$. Let $\alpha_1 > 0$ and $\alpha_2 > 0$, and let $\mu \in \mathbb{R}$. Show that there is a pathwise unique strong solution to the SDE

$$dX_t = \mu X_t \, dt + \alpha_1 X_t \, dB_t^1 + \alpha_2 X_t \, dB_t^2$$

with $X_0 = 1$. Find this solution.

2. Let B be one-dimensional Brownian motion. Suppose $(X_t)_{t \geq 0}$ is a strong solution to the SDE

$$dX_t = X_t^2 \, dt + X_t \, dB_t.$$

with $X_t > 0$ for all t. Show that for all $t \geq 0$, we have

$$X_t = X_0 \exp\left(B_t - B_0 - \frac{t}{2} + \int_0^t X_s \, ds\right).$$

3. Let $(B_t)_{t \geq 0}$ be one-dimensional Brownian motion with $B_0 = 0$. Suppose $b : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ are bounded continuous functions. Suppose $(X_t)_{t \geq 0}$ is a strong solution to the SDE

$$dX_t = \sigma(X_t) \, dB_t + b(X_t) \, dt.$$

Let $a(x) = \sigma(x)^2$ for all x. Let C_b^2 be the set of bounded continuous functions $f : \mathbb{R} \to \mathbb{R}$ whose first two derivatives are also bounded and continuous. For all $f \in C_b^2$, define

$$(Af)(x) = \frac{1}{2} a(x) f''(x) + b(x) f'(x).$$

The operator A defined on C_b^2 is called the *infinitesimal generator* of $(X_t)_{t \geq 0}$.

a) Show that if $f \in C_b^2$ and

$$M_t = f(X_t) - f(X_0) - \int_0^t (Af)(X_s) \, ds$$

for all $t \geq 0$, then $(M_t)_{t \geq 0}$ is a martingale.

b) Show that if $X_0 = x$ and $f \in C_b^2$, then

$$\lim_{t \to 0} \frac{E[f(X_t)] - f(x)}{t} = (Af)(x).$$