Math 10c - Practice Final Solutions

1) Using the u-substitution $u = -0.4x$,

$$\int p(x)dx = \int 0.4e^{-0.4x}dx = -\int e^u du = -e^u + C = -e^{-0.4x} + C$$

We can now use the evaluation theorem to compute the integrals.

(a) $\int_2^3 p(x)dx = (-e^{-0.4\cdot 3}) - (-e^{-0.4\cdot 2}) \approx .15$

(b) $\int_0^2 p(x)dx = (-e^{-0.4\cdot 2}) - (-e^{-0.4\cdot 0}) \approx .55$

(c) $P(t) = \int_0^t p(x)dx = (-e^{-0.4\cdot t}) - (-e^{-0.4\cdot 0}) = 1 - e^{-0.4t}$

2) (a) $\int_{1.5}^2 p(x)dx = \frac{1}{4}\int_{1.5}^2 x^3 dx = \frac{1}{4}\left(\frac{2^4}{4} - \frac{1.5^4}{4}\right) \approx 0.68$

(b) $\int_0^2 x \cdot \frac{x^3}{4}dx = 1.6$ hours.

(c) We solve for T:

$$\frac{1}{2} = \int_0^T \frac{x^3}{4}dx = \frac{1}{16}\left(T^4 - 0^4\right) \Rightarrow 8 = T^4$$

So $T = \sqrt[4]{8} \approx 1.68$ hours.

3) The series is geometric with $a = 3$ and $r = \left(-\frac{1}{2}\right)$. Using the formula $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ for $|r| < 1$, we find that the sum of the series is 2.

4) First compute four derivatives:

$$f'(x) = -\sin x, \quad f''(x) = -\cos x, \quad f^{(3)}(x) = \sin x, \quad f^{(4)}(x) = \cos x$$

Evaluating the function f at $x = 0$ and each of the derivatives as well we obtain

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -1, \quad f^{(3)}(0) = 0, \quad f^{(4)}(0) = 1.$$

The corresponding approximation is $P_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$.

5) A cylinder of radius 3 that is centered about the z-axis.

6) (a) A plane parallel to the yz-plane that passes through the point $(-2, 0, 0)$.
(b) A cylinder of radius 1 centered about the y-axis.

7) Let $c = 2x^2 + y$. Solving for y gives $y = -2x^2 + c$. The level curves will be parabolas. (Be sure to sketch them and label them with the corresponding c value).

8) Substituting $(0, 0, 0)$ into the equation implies $c = 0$. Substituting $(0, 1, 3)$ and solving for n gives $n = 3$. Finally substituting $(-1, -2, 0)$ gives $m = -6$ so that $z = -6x + 3y$.

9) $\|2\vec{i} - 3\vec{j} + \vec{k}\| = \sqrt{14}$. The vector

$$\tfrac{2}{\sqrt{14}}\vec{i} - \tfrac{3}{\sqrt{14}}\vec{j} + \tfrac{1}{\sqrt{14}}\vec{k}$$

is a unit vector. Multiplying this vector by 3 gives the needed vector:

$$\tfrac{6}{\sqrt{14}}\vec{i} - \tfrac{9}{\sqrt{14}}\vec{j} + \tfrac{3}{\sqrt{14}}\vec{k}$$

10) Convert 200m/min to km/hr to obtain 200m/min = 12km/hr. Let $\vec{v} = x_0\vec{i} + y_0\vec{j} + z_0\vec{k}$ be the velocity vector of the airplane. The plane is rising at a rate of 12km/hr so that $z_0 = 12$. The airspeed of the plane is 200 km/hr so $\|\vec{v}\| = 200$. Computing $\|\vec{v}\|$, we get the equation

$$\sqrt{x_0^2 + y_0^2 + 12^2} = 200$$

The x-axis points east, the y-axis points north, and the plane is heading northwest so that $-x_0 = y_0$ (drawing a picture may help you see this). Substituting in for x_0 and solving for y_0 we get $y_0 \approx 141.167$. Therefore the velocity vector of the plane is $\vec{v} = -141.167\vec{i} + 141.167\vec{j} + 12\vec{k}$.

11) $2\vec{v} = 6\vec{i} - 14\vec{j}$ is parallel to $\vec{v}$. If $u = u_1\vec{i} + u_2\vec{j}$, then $\vec{u}$ is perpendicular to $\vec{v}$ if $\vec{u} \cdot \vec{v} = 0$. This gives $3u_1 - 7u_2 = 0$. So $7\vec{i} + 3\vec{j}$ and $-7\vec{i} - 3\vec{j}$ are perpendicular to $\vec{v}$.

12) Find the displacement vectors from P to Q and from P to R:

$$-2\vec{i} + \vec{j} + \vec{k} \qquad \vec{i} + 2\vec{j} + 2\vec{k}$$

Then compute the cross product of these two vectors to get the vector $0\vec{i} + 5\vec{j} - 5\vec{k}$. The equation of the plane is $0 = 0(x - 1) + 5(y - 0) - 5(z - 1)$ which simplifies to $z = y + 1$.

13) $G_s > 0$: Our grade should improve if we increase the number of hours studied. On the other hand, $G_t < 0$: Our grade will go down if we watch more television.

14) (a) $f_x = 2\cos x \cos 3y + y, \qquad f_y = -6\sin x \sin 3y + x$

(b) $z_x = 14xy + 5y$

15) First compute some partial derivatives: $f_x = 2x$ and $f_y = 6y$. Evaluating at $(2,1)$ gives $f_x(2,1) = 4$ and $f_y(2,1) = 6$ so that the equation of the tangent plane is $z = 7 + 4(x-2) + 6(y-1)$.

16) Compute the first partial derivatives:$f_x = \frac{x}{\sqrt{x^2+y^2}}$ and $f_y = \frac{y}{\sqrt{x^2+y^2}}$. Then

$$\nabla f = \frac{x}{\sqrt{x^2+y^2}}\vec{i} + \frac{y}{\sqrt{x^2+y^2}}\vec{j}.$$

17) $\nabla f = 2x\vec{i} + \vec{j}$. Evaluating at $(1,3)$ we get $\nabla f(1,3) = 2\vec{i} + \vec{j}$. An easy computation shows $\|\vec{u}\| = 1$ so $\vec{u}$ is a unit vector. The directional derivative is:

$$\nabla f(1,3) \cdot \vec{u} = 2\left(\tfrac{3}{5}\right) + 1\left(\tfrac{4}{5}\right) = 2.$$

18) $\frac{\partial z}{\partial u} = \frac{e^v}{u}$ and $\frac{\partial z}{\partial v} = (\ln u)e^v + e^v + ve^v$.

19) $f_x = 1+2x+4y, \quad f_y = 4x+8y-1, \quad f_{xx} = 2, \quad f_{yy} = 8, \quad f_{xy} = 4 = f_{yx}$

20) We must compute all first partial derivatives and all second partial derivatives and then evaluate at $(0,0)$:

$$f_x = e^x, \quad f_y = 2e^{2y}, \quad f_{xx} = e^x, \quad f_{xy} = 0, \quad f_{yy} = 4e^{2y}.$$

Now evaluate at $(0,0)$:

$$f(0,0) = 2, \quad f_x(0,0) = 1, \quad f_y(0,0) = 2, \quad f_{xx}(0,0) = 1, \quad f_{xy}(0,0) = 0, \quad f_{yy}(0,0) = 4.$$

The corresponding polynomial is $P_2(x,y) = 2 + x + 2y + \frac{x^2}{2} + 2y^2$.

21) Compute the first and second partial derivatives:

$$f_x = 3x^2 - 6x, \quad f_y = 2y + 10, \quad f_{xx} = 6x - 6, \quad f_{yy} = 2, \quad f_{xy} = 0$$

Solving $f_x = 0$ and $f_y = 0$ gives the critical points $(0,-5)$ and $(2,-5)$.

$D = (6x-2)(2) - 0^2 = 12(x-1)$. At $(0,-5)$, $D < 0$ so there is a saddle point at $(0,-5)$.

At $(2,-5)$, $D > 0$ and $f_{xx} > 0$ so there is a local minimum at $(2,-5)$.

22) There is a global maximum at $(\pm 1, 0)$. The maximum value is 1. There is global minimum at $(0, \pm 1)$ and the minimum value is -1. Both the global maximum and minimum occur on the boundary.

23) Let $g(x, y) = 4x - 2y$. Then $\nabla f = 2x\vec{i} + 2y\vec{j}$ and $\nabla g = 4\vec{i} - 2\vec{j}$. Solving $\nabla f = \lambda \nabla g$ and using the constraint, we obtain the equations:

$$2x = 4\lambda \qquad 2y = -2\lambda \qquad 4x - 2y = 15$$

Multiplying the second by -2 gives $-4y = 4\lambda$ so combining this with the first equation we obtain $2x = -4y$ which gives $x = -2y$. Substituting into the third equation we find $x = 3$ and $y = -\frac{3}{2}$.

We compute $f(3, -\frac{3}{2}) = 11.25$ but we don't know if this is a maximum or a minimum so we compute f at another point that satisfies the constraint and compare. The point $(4, \frac{1}{2})$ satisfies $4x - 2y = 15$ and $f(4, \frac{1}{2}) = 16.25 > 11.25$. So there is a minimum at $(3, -\frac{3}{2})$. There is no maximum.